Hodge Theory for G_2-manifolds: Intermediate Jacobians and Abel-Jacobi maps

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Abstract

We study the moduli space $\mathcal{M}$ of torsion-free G_2-structures on a fixed compact manifold M^7, and define its associated *universal intermediate Jacobian* $\mathcal{J}$. We define the Yukawa coupling and relate it to a natural pseudo-Kähler structure on $\mathcal{J}$.

We consider natural Chern-Simons type functionals, whose critical points give associative and coassociative cycles (calibrated submanifolds coupled with Yang-Mills connections), and also deformed Donaldson-Thomas connections. We show that the moduli spaces of these structures can be isotropically immersed in $\mathcal{J}$ by means of G_2-analogues of *Abel-Jacobi* maps.

1 Introduction

Compact manifolds with G_2 holonomy play the same role in M-theory as Calabi-Yau threefolds do in string theory, both being Ricci-flat and admitting a parallel spinor. In fact, there are analogues of the mirror symmetry phenomenon [1, 11] that are expected to hold in the context of G_2-manifolds. These ideas are still not well understood mathematically.

There are various special geometric structures that can be associated to a G_2-manifold and there are relationships between these structures. Specifically we wish to consider analogues in G_2-geometry of *intermediate Jacobians* and *Abel-Jacobi maps* which are familiar in algebraic geometry.

The moduli space of Calabi-Yau 3-folds is known to admit a *special Kahler structure*. Equivalently, this means that the *universal intermediate Jacobian* $\mathcal{J}$ admits a *hyperKähler structure*. The images in $\mathcal{J}$ of the Abel-Jacobi maps are *Lagrangian submanifolds*. Some references for these facts are [4, 5, 8, 10, 22]. In this paper we prove analogues of these statements for G_2-manifolds. We now describe the organization of the paper, and mention the main results of each section.

In Section 2 we briefly review some well-known facts about G_2-manifolds, which we will need later, and discuss notation.

In Section 3 we begin with the description of the moduli space $\mathcal{M}$ of torsion-free G_2-structures on a fixed G_2-manifold M, which has been studied by Joyce [16, 17] and Hitchin [15]. We define a natural pseudo-Riemannian metric on $\mathcal{M}$ which is the Hessian of a superpotential function f. Finally we define the Yukawa coupling $\mathcal{Y}$ on $\mathcal{M}$, a fully symmetric cubic tensor, and relate this to the superpotential function f.

Using these data, in Section 4 we define intermediate G_2-Jacobians and the universal intermediate G_2-Jacobian $\mathcal{J}$ of the moduli space and show that it admits a natural pseudo-Kähler structure, with Kähler potential (essentially) given by f. Also, the projection $\mathcal{J} \to \mathcal{M}$ is a Lagrangian fibration, and the associated cubic form to this fibration (as described in [8]) is exactly the Yukawa coupling $\mathcal{Y}$.

Section 5 discusses how special geometric structures associated to a G_2-manifold M can be viewed as critical points of certain natural functionals $\Phi_{k,\varphi}$ of Chern-Simons type. Specifically we prove that the calibrated submanifolds of M, together with special connections are such critical points, as are deformed Donaldson-Thomas connections. We study each situation individually.

Additionally, we show that the moduli spaces of such structures can be immersed (via Abel-Jacobi type mappings) into $\mathcal{J}$ isotropically with respect to the appropriate symplectic structure on $\mathcal{J}$. Therefore, these images are Lagrangian subspaces whenever they are half-dimensional. We also discuss a *topological number* $\Psi_{k,\varphi}$ on the moduli spaces of these structures, and relate it to the critical points of $\Phi_{k,\varphi}$.

Finally, in Section 6 we briefly consider the generalization of these results to the case when the cohomology of the G_2-manifold M has torsion, which involves *gerbes*. We also discuss questions for future study.

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2 Review of G_2-structures

In this section we briefly review some facts about G_2-structures. Some references for G_2-structures are [3], [17], [18], [20], [27], and [32]. In addition, the first examples of compact irreducible G_2 manifolds are described in [16] and [17].

Let M^7 be a closed manifold with a G_2-structure. The G_2-structure is given by a positive 3-form φ, which induces an orientation and a Riemannian metric g_φ in a non-linear way via the formula

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6\, g_\varphi(X, Y)\, \mathsf{vol}_\varphi$$

where vol_φ is the volume form corresponding to the orientation and the metric g_φ. Hence there is an induced Hodge star operator $*_\varphi$ and the associated dual 4-form $\psi = *_\varphi \varphi$. The 3-form has constant pointwise norm $|\varphi|^2 = 7$ with respect to g_φ.

The G_2-structure is called *torsion-free* if φ is parallel with respect to its induced metric g_φ. In this case the Riemannian holonomy is contained in G_2, and M^7 is called a G_2-*manifold*. It is well

known that a G_2-structure φ is torsion-free if and only if φ is both closed and co-closed (with respect to g_φ.)

The space of forms Ω^* on a manifold with G_2-structure decomposes into G_2-representations, and this decomposition descends to the cohomology in the torsion-free case. Specifically, the cohomology breaks up as

$$\begin{aligned} H^2(M,\mathbb{R}) &= H^2_7 \oplus H^2_{14} \\ H^3(M,\mathbb{R}) &= H^3_1 \oplus H^3_7 \oplus H^3_{27} \\ H^4(M,\mathbb{R}) &= H^4_1 \oplus H^4_7 \oplus H^4_{27} \\ H^5(M,\mathbb{R}) &= H^5_7 \oplus H^5_{14} \end{aligned}$$

We define $b^k_l = \dim(H^k_l)$. Then $b^3_1 = b^4_1 = 1$, and $b^k_7 = b_1(M)$, for $k = 2,3,4,5$. The actual holonomy group $\mathrm{Hol}(g_\varphi)$ of (M, g_φ) is determined by the topology of M. For example, the holonomy is exactly G_2 if and only if $\pi_1(M)$ is finite, in which case $b_1(M) = 0$ and all the $b^k_7 = 0$. Such an M^7 is called *irreducible*.

A result which we will need repeatedly is the following.

Lemma 2.1. *Let $\varphi_t = \varphi_0 + t\eta$ be a one-parameter family of G_2-structures for small t. Then we have*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (*_{\varphi_t}\varphi_t) = \frac{4}{3} *_{\varphi_0} \pi_1(\eta) + *_{\varphi_0}\pi_7(\eta) - *_{\varphi_0}\pi_{27}(\eta) \tag{2.1}$$

where $*_{\varphi_0}$ is the Hodge star induced from φ_0, and π_k is the orthogonal projection onto Ω^3_k (defined with respect to φ_0.)

Proof. This is first mentioned in [16], and explicit proofs can be found in [15] and [20]. □

It follows from Lemma 2.1 that if $\psi_t = \psi_0 + t\theta$ is a one-parameter family of postive 4-forms, then

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (*_{\psi_t}\psi_t) = \frac{3}{4} *_{\varphi_0} \pi_1(\theta) + *_{\varphi_0}\pi_7(\theta) - *_{\varphi_0}\pi_{27}(\theta)$$

which motivates the following definition.

Definition 2.2. Let φ_0 be a fixed torsion-free G_2-structure. We define the map $\star_{\varphi_0} : \Omega^k \to \Omega^{7-k}$ for $k = 3, 4$ by

$$\text{for } \eta \in \Omega^3; \qquad \star_{\varphi_0}(\eta) = \frac{4}{3} *_{\varphi_0} \pi_1(\eta) + *_{\varphi_0}\pi_7(\eta) - *_{\varphi_0}\pi_{27}(\eta) \tag{2.2}$$

$$\text{for } \theta \in \Omega^4; \qquad \star_{\varphi_0}(\theta) = \frac{3}{4} *_{\varphi_0} \pi_1(\theta) + *_{\varphi_0}\pi_7(\theta) - *_{\varphi_0}\pi_{27}(\theta) \tag{2.3}$$

Notice that $\star_{\varphi_0}$ agrees (up to a constant) with $*_{\varphi_0}$ on each Ω^k_l, but the constants are different on different components. Also note that $\star^2_{\varphi_0} = 1$.

Finally we make some remarks about notation. We use $\langle \alpha, \beta \rangle$ and $|\alpha|^2$ to denote the pointwise inner product and norm on forms induced from g_φ. We use $\langle\langle \alpha, \beta \rangle\rangle = \int_M \langle \alpha, \beta \rangle \mathsf{vol}_\varphi$ and $\|\alpha\|^2 = \langle\langle \alpha, \alpha \rangle\rangle$ to denote the corresponding L^2 inner product.

As M is always taken to be compact, we use Hodge theory throughout, so we will often identify a cohomology class $[\alpha]$ with its unique harmonic representative α. We use $g = g_\varphi$ to denote the Riemannian metric on M^7 associated to the G_2-structure φ, and reserve $\mathcal{G}$ for the metric on the

moduli space $\mathcal{M}$ of G_2-manifolds, and $\mathcal{G}_{\mathcal{J}}$ for the pseudo-Kähler metric on the universal intermediate Jacobian $\mathcal{J}$. The summation convention is used throughout, and the dimension of the moduli space $\mathcal{M}$ is $b_3 = n + 1$ with indices running from 0 to n.

3 Special Geometry of the G_2-moduli space

3.1 The moduli space of G_2-manifolds

We denote by $\mathcal{M}$ the moduli space of torsion-free G_2-structures on M. Explicitly, $\mathcal{M}$ is defined as

$$\mathcal{M} = \left\{\varphi \in \Omega^3_+(M) : \nabla_\varphi \varphi = 0\right\} / \mathrm{Diff}_0(M)$$

where $\Omega^3_+(M)$ denotes the space of G_2-structures on M (also called the *positive* 3-forms), and $\mathrm{Diff}_0(M)$ is the space of diffeomorphisms of M isotopic to the identity. This should more properly be called the *Teichmuller space* of G_2-structures on M.

Remark 3.1. The quotient of the torsion-free G_2-structures by the *full* diffeomorphism group of M (which is equivalent to the quotient of $\mathcal{M}$ by the mapping class group of M) is in general an *orbifold*. As most of our discussions are local, we could equally consider this moduli space, at least where it is smooth.

Dominic Joyce proved in [16, 17] that $\mathcal{M}$ is a smooth manifold of dimension $b_3(M)$. Note that on a compact G_2-manifold, $b_3 \geq 1$, since φ is a non-trivial harmonic 3-form. Therefore the moduli space is always at least one-dimensional, and those moduli correspond to constant scalings of φ, and hence scalings of the total volume of M. In [16, 17], Joyce also proved the following local result.

Theorem 3.2 (Joyce [16, 17])**.** *The period map*

$$\begin{aligned} P \quad &: \quad \mathcal{M} \to H^3(M, \mathbb{R}) \\ P(\varphi) \quad &= \quad [\varphi] \end{aligned}$$

is a local diffeomorphism. Here $[\varphi]$ denotes the cohomology class of φ. Hence, given $\varphi_o \in \mathcal{M}$, there exists $\epsilon > 0$ such that

$$\mathcal{U}_{\varphi_o} = \left\{\varphi \in \Omega^3_+(M) : \|\varphi - \varphi_o\|_{g_{\varphi_o}} < \epsilon, \nabla_\varphi \varphi = 0\right\} / \mathrm{Diff}_0(M)$$

is diffeomorphic to an open set in the vector space $H^3(M, \mathbb{R})$.

This result says $\mathcal{M}$ has a natural *affine* structure and a *flat connection* ∇, and P is an affine map. To see this, let $\eta_0, \ldots, \eta_n$ be a basis of $H^3(M, \mathbb{R})$, where $n + 1 = b_3 = b^3_7 + b^3_{27} + 1$. Let x^i be coordinates on $\mathcal{U}_{\varphi_0}$ with respect to this basis. Thus $\eta = x^i \eta_i$ describes a point in $\mathcal{U}_{\varphi_0}$ and $\eta_i = \frac{\partial}{\partial x^i}$. Since $\mathcal{U}_{\varphi_0}$ is an open set in a vector space, it admits a flat connection $\nabla = d$, and because $\nabla(dx^i) = d(dx^i) = 0$, we say x^i are *flat* coordinates. We can thus cover $\mathcal{M}$ by such flat charts.

On two overlapping flat charts, the coordinate systems $\tilde{x}^i$ and x^i are affinely related: $x^i = P^i_j \tilde{x}^j + Q^i$, where P^i_j and Q^i are constants. Thus $\frac{\partial}{\partial \tilde{x}^j} = P^i_j \frac{\partial}{\partial x^i}$, and the transition functions are constant. Hence the flat connection ∇ is a well-defined connection on $\mathcal{M}$, since if $\nabla = d + A$ with $A = 0$ in one flat chart, then $\tilde{A} = P^{-1} A P + P^{-1} dP = 0$ in all overlapping flat charts. Such structures also arise in [12, 13, 26, 28], for example.

There is a natural *pseudo-Riemannian* metric $\mathcal{G}$ on the moduli space $\mathcal{M}$ first defined by Hitchin [15] which we now describe.

Definition 3.3. We define a smooth function $f : \mathcal{M} \to \mathbb{R}$ as follows:

$$f(\varphi) = \frac{3}{7}\int_M (\varphi \wedge *_\varphi \varphi) = 3\int_M \mathsf{vol}_\varphi \tag{3.1}$$

where φ is a point in $\mathcal{M}$. Thus, up to a constant, $f([\varphi])$ is the total volume of M with respect to the metric and orientation induced by φ. We call f the *superpotential function* for $\mathcal{M}$, for reasons that will soon be clear.

Remark 3.4. Our definition differs from Hitchin's by a constant factor, which we choose for later convenience.

In [15], Hitchin shows that when restricted to closed G_2-structures in a fixed cohomology class, the torsion-free G_2-structures are precisely those which are critical points of f. We will say more about this in Section 5.4.

Theorem 3.5 (Hitchin [15])**.** *In a flat coordinate chart $(\mathcal{U}_{\varphi_0}, x^0, \ldots, x^n)$, the Hessian $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ defines a pseudo-Riemannian metric $\mathcal{G}$ on $\mathcal{M}$ by $\mathcal{G}_{ij} = f_{ij}$. In the case when* $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$*, the metric $\mathcal{G}_{ij}$ is* Lorentzian.

Proof. Let φ be a point in $\mathcal{M}$. We want to differentiate f in the η_j direction. We identify the cohomology class η_j with its unique harmonic representative (with respect to the metric g_φ induced from φ.) Now by Lemma 2.1, we have

$$\begin{aligned}
\frac{\partial f}{\partial x^j} &= \frac{3}{7}\int_M \left(\frac{\partial}{\partial x^j}\varphi\right) \wedge *_\varphi \varphi + \frac{3}{7}\int_M \varphi \wedge \left(\frac{\partial}{\partial x^j} *_\varphi \varphi\right) \\
&= \frac{3}{7}\int_M \eta_j \wedge *_\varphi \varphi + \frac{3}{7}\int_M \varphi \wedge \left(\frac{4}{3} *_\varphi \pi_1(\eta_j) + *_\varphi \pi_7(\eta_j) - *_\varphi \pi_{27}(\eta_j)\right) \\
&= \frac{3}{7}\int_M \pi_1(\eta_j) \wedge *_\varphi \varphi + \frac{4}{7}\int_M \varphi \wedge *_\varphi \pi_1(\eta_j) \\
&= \int_M \pi_1(\eta_j) \wedge *_\varphi \varphi = \int_M \eta_j \wedge *_\varphi \varphi
\end{aligned}$$

where we have used the fact that the splitting of Ω^3 is orthogonal with respect to g_φ. To compute the second derivative, let $\varphi(t)$ be a one-parameter family of torsion-free G_2-structures with $\varphi(0) = \varphi$, satisfying $\frac{\partial}{\partial t}\varphi(t) = \eta_i(t)$, where $\eta_i(t)$ is the unique harmonic representative of the cohomology class η_i with respect to the metric g_t induced from $\varphi(t)$. Then $\eta_j(t) = \eta_j(0) + d\beta(t)$ for some smooth family of 2-forms $\beta(t)$. Hence $\frac{\partial}{\partial t}\eta_j(t) = d\beta'(t)$ is exact.

Writing $\eta_j(0) = \eta_j$, we compute the second derivative at φ as

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^i \partial x^j} &= \int_M \left(\frac{\partial}{\partial t}\bigg|_{t=0} \eta_j(t)\right) \wedge *_\varphi \varphi + \int_M \eta_j \wedge \left(\frac{\partial}{\partial t}\bigg|_{t=0} *_{\varphi(t)} \varphi(t)\right) \\
&= \int_M d\beta'(0) \wedge *_\varphi \varphi + \int_M \eta_j \wedge \left(\frac{4}{3} *_\varphi \pi_1(\eta_i) + *_\varphi \pi_7(\eta_i) - *_\varphi \pi_{27}(\eta_i)\right) \\
&= \int_M \left(\frac{4}{3}\pi_1(\eta_i) \wedge *_\varphi \pi_1(\eta_j) + \pi_7(\eta_i) \wedge *_\varphi \pi_7(\eta_j) - \pi_{27}(\eta_i) \wedge *_\varphi \pi_{27}(\eta_j)\right) \\
&= \frac{4}{3}\langle\!\langle \pi_1(\eta_i), \pi_1(\eta_j) \rangle\!\rangle + \langle\!\langle \pi_7(\eta_i), \pi_7(\eta_j) \rangle\!\rangle - \langle\!\langle \pi_{27}(\eta_i), \pi_{27}(\eta_j) \rangle\!\rangle
\end{aligned}$$

where the first term in the second line above vanishes by Stokes' theorem since $*_\varphi \varphi$ is closed and M is compact.

The Laplace and Green's operators commute with the projections when φ is torsion-free, so each $\pi_k(\eta_j)$ is harmonic. Therefore we see that f_{ij} is a pseudo-Riemannian metric of signature $(b^3_1 + b^3_7, b^3_{27}) = (1 + b_1, b^3_{27})$.

When M is irreducible (so the holonomy is exactly G_2), we know $b_1 = 0$. Since $H^1 \cong H^3_7$, there are no harmonic Ω^3_7-forms, and thus in this case

$$f_{ij} = \frac{4}{3}\langle\langle \pi_1(\eta_i), \pi_1(\eta_j)\rangle\rangle - \langle\langle \pi_{27}(\eta_i), \pi_{27}(\eta_j)\rangle\rangle \tag{3.2}$$

which is a Lorentzian metric of signature $(1, b^3_{27})$. □

Remark 3.6. We defined this metric using a particular flat coordinate chart in a neighbourhood of each point. However, this metric $\mathcal{G}$ is well-defined globally on $\mathcal{M}$. This is because with respect to any other overlapping flat coordinate chart $(\tilde{x}^0, \ldots \tilde{x}^n)$, the two sets of coordinates are affinely related: $x^i = P^i_j \tilde{x}^j + Q^i$. Therefore $\frac{\partial}{\partial \tilde{x}^k} = P^i_k \frac{\partial}{\partial x^i}$, and also $\frac{\partial^2 f}{\partial \tilde{x}^k \partial \tilde{x}^l} = P^i_k P^j_l \frac{\partial^2 f}{\partial x^i \partial x^j}$. This is exactly the statement that the metric $\mathcal{G}$ is well-defined.

Remark 3.7. In the physics literature, a slightly different notion of superpotential function is used: they define $F = -\log(f)$ and use the Hessian of F to define a metric on $\mathcal{M}$. The advantage of this definition is that the metric F_{ij} is actually *Riemannian* when M is irreducible, as opposed to Lorentzian. However, with this choice of potential function and metric, other results fail to hold. See Remarks 3.13, 4.9, and 5.8 for more details. It is for these reasons that we prefer to use f and its associated Lorentzian metric.

We close this section by noting that we can write the metric $\mathcal{G}$ on $\mathcal{M}$ more concisely using the operator $\star_\varphi$ defined in (2.2) as

$$\mathcal{G}(\eta_1, \eta_2) = \int_M \eta_1 \wedge \star_\varphi \eta_2 \tag{3.3}$$

for $\eta_i \in T_\varphi \mathcal{M} = H^3(M, \mathbb{R})$. We stress again that $\star_\varphi$ is not the same as $*_\varphi$, and thus $\mathcal{G}$ is not the usual L^2 metric on forms. This metric $\mathcal{G}$ is more natural than the usual L^2 metric, because the fact that it is of Hessian type allows us to construct a pseudo-Kähler structure on the universal intermediate Jacobian in Section 4.2.

3.2 The Yukawa Coupling

In this section we define the *Yukawa coupling* $\mathcal{Y}$, a symmetric cubic tensor on the G_2-moduli space $\mathcal{M}$. We relate this tensor to the superpotential function f and the metric $\mathcal{G}$ on $\mathcal{M}$. We will need the following identity, which can be found in Lemma A.12 of [20]:

$$\varphi_{ijk}\varphi_{abc}g^{kc} = g_{ia}g_{jb} - g_{ib}g_{ja} - \psi_{ijab} \tag{3.4}$$

We recall some facts about the space $\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27}$ of 3-forms on a G_2-manifold. We adopt the notation of [20]. The elements $\eta \in \Omega^3_1 \oplus \Omega^3_{27}$ are in one-to-one correspondence with symmetric 2-tensors $h_{ij} \in S^2(T^*)$. The correspondence is given by

$$\eta_{ijk} = h_{il}g^{lm}\varphi_{mjk} + h_{jl}g^{lm}\varphi_{imk} + h_{kl}g^{lm}\varphi_{ijm} \tag{3.5}$$

where Ω^3_1 corresponds to multiples of the metric g and Ω^3_{27} corresponds to the *traceless* symmetric tensors. Note that the 3-form φ itself corresponds to the symmetric tensor $\frac{1}{3}g_{ij}$.

From now on we assume that M is irreducible, so $H^3(M,\mathbb{R}) = H^3_1 \oplus H^3_{27}$. Each cohomology class has a unique harmonic representative, so for a fixed torsion-free G_2-structure φ with metric g_φ, each class $\eta_k \in H^3(M,\mathbb{R})$ corresponds to a symmetric 2-tensor h_k.

Definition 3.8. The *Yukawa coupling* $\mathcal{Y}$ is a fully symmetric cubic tensor on the moduli space $\mathcal{M}$, defined as follows. Let η_i, η_j, η_k be elements of $T_\varphi\mathcal{M} \cong H^3(M,\mathbb{R})$, with associated symmetric tensors h_i, h_j, h_k, with respect to g_φ. Then we define

$$\mathcal{Y}(\eta_i,\eta_j,\eta_k) = \int_M (h_i)^{a\alpha}(h_j)^{b\beta}(h_k)^{c\gamma}\phi_{abc}\phi_{\alpha\beta\gamma}\mathsf{vol} \tag{3.6}$$

where g and vol are with respect to φ, and $(h_i)^{a\alpha}$ means $(h_i)_{b\beta}g^{ba}g^{\beta\alpha}$, raising indices with g. It is clear that $\mathcal{Y}$ is fully symmetric in its arguments.

In Section 4.2, we will see that the Yukawa coupling $\mathcal{Y}$ is essentially the natural cubic form associated to a Lagrangian fibration.

Fix φ in $\mathcal{M}$. Let $\eta_0, \eta_1, \ldots, \eta_n$ be a basis for $T_\varphi\mathcal{M} \cong H^3(M,\mathbb{R})$, where $\eta_0 = \varphi$ and $\eta_i \in H^3_{27}$ for $i \neq 0$.

Proposition 3.9. *The following relations between the Yukawa coupling $\mathcal{Y}$, the superpotential function f, and the Hessian metric $\mathcal{G}_{ij} = f_{ij}$ hold:*

$$\begin{aligned}
\mathcal{Y}(\varphi,\varphi,\varphi) &= \frac{14}{27}\,f(\varphi) &(3.7)\\
\mathcal{Y}(\varphi,\varphi,\eta_i) &= 0 &(3.8)\\
\mathcal{Y}(\varphi,\eta_i,\eta_j) &= \frac{1}{6}\,\mathcal{G}_{ij}(\varphi) &(3.9)
\end{aligned}$$

where $i,j = 0,\ldots,n$, and $i \neq 0$ in (3.8).

Proof. First we show that (3.7) and (3.8) follow from (3.9). Equation (3.8) is automatic since $\mathcal{G}_{0i} = 0$ for $i \neq 0$ by (3.2). Also (3.2) shows $\mathcal{G}_{00} = \frac{4}{3}\int \varphi \wedge *\varphi = \frac{28}{9}f(\varphi)$. Then (3.9) says $\mathcal{Y}(\varphi,\varphi,\varphi) = \frac{1}{6}\mathcal{G}_{00} = \frac{14}{27}f(\varphi)$.

To establish (3.9), consider $h_k = \frac{1}{3}g$ in (3.6) and use (3.4). We obtain:

$$\begin{aligned}
\mathcal{Y}(\varphi,\eta_i,\eta_j) &= \frac{1}{3}\int_M (h_i)^{a\alpha}(h_j)^{b\beta}\varphi_{abc}\varphi_{\alpha\beta\gamma}g^{c\gamma}\mathsf{vol}\\
&= \frac{1}{3}\int_M (h_i)^{a\alpha}(h_j)^{b\beta}\left(g_{a\alpha}g_{b\beta} - g_{a\beta}g_{b\alpha} - \psi_{ab\alpha\beta}\right)\mathsf{vol}\\
&= \frac{1}{3}\int_M \left(\mathrm{Tr}(h_i)\,\mathrm{Tr}(h_j) - \mathrm{Tr}(h_ih_j)\right)\mathsf{vol} &(3.10)
\end{aligned}$$

where $(h_ih_j)_{ab} = (h_i)_{al}g^{lm}(h_j)_{mb}$ denotes matrix multiplication. The third term vanished by the symmetry of the h's and the skew-symmetry of ψ. Let $h_i = \lambda g + h^0_i$ and $h_j = \mu g + h^0_j$ where h^0_i and h^0_j are the trace-free parts. Substituting these into (3.10) gives

$$\mathcal{Y}(\varphi,\eta_i,\eta_j) = \frac{1}{3}\int_M \left(42\,\lambda\mu - \mathrm{Tr}(h^0_ih^0_j)\right)\mathsf{vol} \tag{3.11}$$

In Proposition 2.15 of [20], it is shown that for $\eta_i, \eta_j \in \Omega^3_1 \oplus \Omega^3_{27}$, we have

$$\begin{aligned}\int_M (\eta_i \wedge *\eta_j)\mathsf{vol} &= \int_M (\mathrm{Tr}(h_i)\,\mathrm{Tr}(h_j) + 2\,\mathrm{Tr}(h_i h_j))\,\mathsf{vol} \\ &= \int_M \left(63\,\lambda\mu + 2\,\mathrm{Tr}(h^0_i h^0_j)\right)\mathsf{vol} \end{aligned} \tag{3.12}$$

By (3.2), the metric $\mathcal{G}_{ij} = f_{ij}$ is given by

$$\mathcal{G}_{ij} = \int_M \eta_i \wedge *\bar{\eta}_j \mathsf{vol}$$

where $\bar{\eta}_j$ corresponds to the symmetric tensor $\bar{h}_j = \frac{4}{3}\mu g - h^0_j$. Substituting $\mu \to \frac{4}{3}\mu$ and $h^0_j \to -h^0_j$ in (3.12) shows

$$\mathcal{G}_{ij} = \int_M \left(84\,\lambda\mu - 2\,\mathrm{Tr}(h^0_i h^0_j)\right)\mathsf{vol} \tag{3.13}$$

Comparing (3.11) and (3.13), shows $\mathcal{Y}(\varphi, \eta_i, \eta_j) = \frac{1}{6}\,\mathcal{G}_{ij}$. □

From the above proposition, the natural question which arises is whether $\mathcal{Y}(\eta_i, \eta_j, \eta_k)$ is related to f_{ijk}. The main theorem of this section is the following, which shows that this is indeed the case.

Theorem 3.10. *Let η_i, η_j, η_k be in $T_\varphi\mathcal{M} \cong H^3(M, \mathbb{R})$. Then*

$$\mathcal{Y}(\eta_i, \eta_j, \eta_k) = \frac{1}{2}\, f_{ijk} = \frac{1}{2}\, \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k} \tag{3.14}$$

for $i, j, k = 0, \ldots, n$.

Before we can prove this theorem, we need a couple of preliminary results. The reason is that it is difficult to directly differentiate $f_{ij} = \int_M \eta_i \wedge \star_\varphi \eta_j$ because of the complicated nature of the operator $\star_\varphi$. We circumvent this difficulty by considering the auxiliary function $F = -\log(f)$ instead, because it will turn out that the third derivative of F is easily computable.

Lemma 3.11. *Let $F = -\log(f)$. Then the Hessian $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$ is given by*

$$F_{ij} = \frac{1}{f}\int_M \eta_i \wedge *_\varphi \eta_j = \frac{1}{f}\langle\!\langle \eta_i, \eta_j \rangle\!\rangle \tag{3.15}$$

when M is irreducible.

Proof. In this proof, subscripts on f or F always denote partial differentiation. We evaluate all derivatives at a fixed point φ in $\mathcal{M}$. As before fix a basis $\eta_0, \ldots, \eta_n$ of $H^3(M, \mathbb{R})$ so that $\eta_0 = \varphi$. From the proof of Theorem 3.5, and our choice of basis, we see that at the point φ in $\mathcal{M}$,

$$f_0 = \frac{7}{3} f \qquad\qquad f_i = 0 \quad \text{for } i \neq 0 \tag{3.16}$$

and also that

$$f_{00} = \frac{28}{9} f \qquad f_{i0} = 0 \qquad f_{ij} = -\int \langle \eta_i, \eta_j \rangle \mathsf{vol} \quad \text{for } i, j \neq 0 \tag{3.17}$$

From $F = -\log(f)$ we have $F_i = -f^{-1} f_i$ and

$$F_{ij} = \frac{1}{f^2} f_i f_j - \frac{1}{f} f_{ij} \tag{3.18}$$

Substituting the above expressions, we see that at the point φ in $\mathcal{M}$, we have

$$F_{00} = \frac{7}{3} = \frac{1}{f} \int \varphi \wedge *_\varphi \varphi \qquad F_{i0} = 0 \qquad F_{ij} = +\frac{1}{f} \int \eta_i \wedge *_\varphi \eta_j \quad \text{for } i, j \neq 0$$

and hence in all cases we get

$$F_{ij} = \frac{1}{f} \int_M \eta_i \wedge *_\varphi \eta_j = \frac{1}{f} \langle\!\langle \eta_i, \eta_j \rangle\!\rangle \qquad i, j = 0, \ldots, n$$

as claimed. □

Lemma 3.12. *Let η_1, η_2, and η_3 be 3-forms in $\Omega^3_1 \oplus \Omega^3_{27}$, with associated symmetric tensors h_1, h_2, and h_3, respectively. Then*

$$\begin{aligned} \int_M (\eta_1)_{ijk} (\eta_2)_{abc} (h_3)^{ia} g^{jb} g^{kc} \mathsf{vol} \;&=\; 2\, \mathcal{Y}(\eta_1, \eta_2, \eta_3) \\ &\quad + 2 \int_M \left(\mathrm{Tr}(h_1)\, \mathrm{Tr}(h_2 h_3) + \mathrm{Tr}(h_2)\, \mathrm{Tr}(h_3 h_1) + \mathrm{Tr}(h_3)\, \mathrm{Tr}(h_1 h_2) \right) \mathsf{vol} \end{aligned} \tag{3.19}$$

Proof. We use (3.5) and exploit the symmetry of h_1, h_2, h_3, g, and the skew-symmetry of φ to compute:

$$\begin{aligned} (\eta_1)_{ijk} (\eta_2)_{abc} (h_3)^{ia} g^{jb} g^{kc} \;&=\; (h_1)_{il} g^{lm} \varphi_{mjk} (\eta_2)_{abc} (h_3)^{ia} g^{jb} g^{kc} \\ &\quad + 2\, (h_1)_{jl} g^{lm} \varphi_{imk} (\eta_2)_{abc} (h_3)^{ia} g^{jb} g^{kc} \end{aligned}$$

and then

$$\begin{aligned} (\eta_1)_{ijk} (\eta_2)_{abc} (h_3)^{ia} g^{jb} g^{kc} \;&=\; (h_1)_{il} (h_2)_{a\alpha} g^{lm} g^{\alpha\beta} \varphi_{mjk} \varphi_{\beta bc} (h_3)^{ia} g^{jb} g^{kc} \\ &\quad + 2\, (h_1)_{il} (h_2)_{b\alpha} g^{lm} g^{\alpha\beta} \varphi_{mjk} \varphi_{a\beta c} (h_3)^{ia} g^{jb} g^{kc} \\ &\quad + 2\, (h_1)_{jl} (h_2)_{a\alpha} g^{lm} g^{\alpha\beta} \varphi_{imk} \varphi_{\beta bc} (h_3)^{ia} g^{jb} g^{kc} \\ &\quad + 2\, (h_1)_{jl} (h_2)_{b\alpha} g^{lm} g^{\alpha\beta} \varphi_{imk} \varphi_{a\beta c} (h_3)^{ia} g^{jb} g^{kc} \\ &\quad + 2\, (h_1)_{jl} (h_2)_{c\alpha} g^{lm} g^{\alpha\beta} \varphi_{imk} \varphi_{ab\beta} (h_3)^{ia} g^{jb} g^{kc} \end{aligned}$$

We use the identity (3.4) repeatedly, and simplify. The right hand side becomes

$$\begin{aligned} &6\ \mathrm{Tr}(h_1 h_2 h_3) + 2\, (\mathrm{Tr}(h_2)\, \mathrm{Tr}(h_3 h_1) - \mathrm{Tr}(h_2 h_3 h_1)) \\ &+ 2\, (\mathrm{Tr}(h_1)\, \mathrm{Tr}(h_2 h_3) - \mathrm{Tr}(h_1 h_2 h_3)) + 2\, (\mathrm{Tr}(h_3)\, \mathrm{Tr}(h_1 h_2) - \mathrm{Tr}(h_3 h_1 h_2)) \\ &+ 2 (h_1)^{ia} (h_2)^{jb} (h_3)^{kc} \varphi_{ijk} \varphi_{abc} \end{aligned}$$

which, upon further simplification and integration over M, is exactly (3.19). □

Proof of Theorem 3.10. We begin by differentiating (3.18) with respect to x^k and rearranging to obtain

$$f_{ijk} = -fF_{ijk} - \frac{2}{f^2} f_i f_j f_k + \frac{1}{f} f_i f_{jk} + \frac{1}{f} f_j f_{ki} + \frac{1}{f} f_k f_{ij} \tag{3.20}$$

We need to compute F_{ijk}. Using Lemma 3.11, this is

$$\begin{aligned} F_{ijk} &= -\frac{1}{f^2} f_k \int_M \eta_i \wedge *_\varphi \eta_j + \frac{1}{f} \int_M \left(\frac{\partial}{\partial x^k} \eta_i \right) \wedge *_\varphi \eta_j \\ &\quad + \frac{1}{f} \int_M \left(\frac{\partial}{\partial x^k} \eta_j \right) \wedge *_\varphi \eta_i + \frac{1}{f} \int_M \eta_i \wedge \left(\frac{\partial}{\partial x^j} *_\varphi \right) \eta_j \end{aligned} \tag{3.21}$$

As in the proof of Theorem 3.5, we have $\frac{\partial}{\partial x^k} \eta_i = d\beta_i'$ is exact. Since the η_i's are harmonic with respect to g_φ, we have that $*_\varphi \eta_i$ is closed, and hence by Stokes' theorem the second and third terms above both vanish. The fourth term of (3.21) is

$$\frac{1}{f} \int_M \frac{\partial}{\partial x^k} \left(g_\varphi(\eta_i, \eta_j) \right) \mathsf{vol} + \frac{1}{f} \int_M g_\varphi(\eta_i, \eta_j) \frac{\partial}{\partial x^k} \mathsf{vol} \tag{3.22}$$

where we regard η_i and η_j as constant. In local coordinates, we have

$$g_\varphi(\eta_i, \eta_j) = \frac{1}{6} (\eta_i)_{abc} (\eta_j)_{\alpha\beta\gamma} g^{a\alpha} g^{b\beta} g^{c\gamma}$$

We are differentiating in the x^k direction, which means that we are considering a one-parameter family $\varphi(t)$ of torsion-free G_2-structures, such that $\frac{\partial}{\partial t} \varphi(t) = \eta_k(t)$, which at $t = 0$ corresponds to a symmetric 2-tensor h_k. From Corollary 3.3 of [20], such an infinitesmal variation induces

$$\frac{\partial}{\partial x^k} g^{ab} = -2 \, (h_k)^{ab} \qquad\qquad \frac{\partial}{\partial x^k} \mathsf{vol} = \mathrm{Tr}(h_k) \, \mathsf{vol}$$

as the variations of g^{ab} and vol, respectively. Substituting these results and simplifying expression (3.22), equation (3.21) becomes

$$\begin{aligned} F_{ijk} &= -\frac{1}{f^2} f_k \int_M \eta_i \wedge *_\varphi \eta_j + \frac{1}{f} \int_M \mathrm{Tr}(h_k) \, \eta_i \wedge *_\varphi \eta_j \\ &\quad + \frac{1}{f} \int_M \left(-(\eta_i)_{abc} (\eta_j)_{\alpha\beta\gamma} (h_k)^{a\alpha} g^{b\beta} g^{c\gamma} \right) \mathsf{vol} \end{aligned} \tag{3.23}$$

Now there are two cases. If $k \neq 0$, then $\mathrm{Tr}(h_k) = 0$ and $f_k = 0$ from (3.16). Whereas if $k = 0$, then $h_0 = \frac{1}{3} g$, so $\mathrm{Tr}(h_0) = \frac{7}{3}$, and $f_0 = \frac{7}{3} f$ by (3.16). Thus in all cases the combination of the first two terms of (3.23) vanishes. Now applying Lemma 3.12, we finally obtain:

$$\begin{aligned} -fF_{ijk} &= 2 \, \mathcal{Y}(\eta_1, \eta_2, \eta_3) \\ &\quad + 2 \int_M \left(\mathrm{Tr}(h_i) \, \mathrm{Tr}(h_j h_k) + \mathrm{Tr}(h_j) \, \mathrm{Tr}(h_k h_i) + \mathrm{Tr}(h_k) \, \mathrm{Tr}(h_i h_j) \right) \mathsf{vol} \end{aligned}$$

Substituting this into (3.20) gives

$$\begin{aligned} f_{ijk} &= 2 \, \mathcal{Y}(\eta_1, \eta_2, \eta_3) - \frac{2}{f^2} f_i f_j f_k + \frac{1}{f} f_i f_{jk} + \frac{1}{f} f_j f_{ki} + \frac{1}{f} f_k f_{ij} \\ &\quad + 2 \int_M \left(\mathrm{Tr}(h_i) \, \mathrm{Tr}(h_j h_k) + \mathrm{Tr}(h_j) \, \mathrm{Tr}(h_k h_i) + \mathrm{Tr}(h_k) \, \mathrm{Tr}(h_i h_j) \right) \mathsf{vol} \end{aligned} \tag{3.24}$$

Case 1: $i = j = k = 0$. Then since $h_0 = \frac{1}{3}g$, we have $\mathrm{Tr}(h_0) = \frac{7}{3}$ and $\mathrm{Tr}(h_0 h_0) = \frac{7}{9}$, and $f_0 = \frac{7}{3}f$ and $f_{00} = \frac{28}{9}f$ by (3.16) and (3.17). Thus (3.24) is

$$\begin{aligned} f_{000} &= 2\,\mathcal{Y}(\eta_0, \eta_0, \eta_0) - \frac{2}{f^2}\left(\frac{7}{3}f\right)^3 + \frac{3}{f}\left(\frac{7}{3}f\right)\left(\frac{28}{9}f\right) + 6\left(\frac{7}{3}\right)\left(\frac{7}{9}\right)\int_M \mathsf{vol} \\ &= 2\,\mathcal{Y}(\eta_0, \eta_0, \eta_0) - \frac{686}{27}f + \frac{196}{9}f + \frac{294}{81}f = 2\,\mathcal{Y}(\eta_0, \eta_0, \eta_0) \end{aligned}$$

where we have used $f = 3\int \mathsf{vol}$.

Case 2: $j = k = 0$, $i \neq 0$. We have $\mathrm{Tr}(h_i) = 0$ and $\mathrm{Tr}(h_i h_0) = 0$. Also $f_i = 0$ and $f_{i0} = 0$ by (3.16) and (3.17). Thus all the extra terms vanish and (3.24) in this case says

$$f_{i00} = 2\,\mathcal{Y}(\eta_i, \eta_0, \eta_0)$$

Case 3: $k = 0$, $i, j \neq 0$. In this case the only non-vanishing terms on the right hand side of (3.24), besides the $\mathcal{Y}$ term, are the last terms on each line. This time we get

$$f_{ij0} = 2\,\mathcal{Y}(\eta_i, \eta_j, \eta_0) + \frac{7}{3}f_{ij} + \frac{14}{3}\int_M \mathrm{Tr}(h_i h_j)\,\mathsf{vol}$$

Since h_i and h_j are traceless, equation (3.13) shows that the last two terms cancel, and hence

$$f_{ij0} = 2\,\mathcal{Y}(\eta_i, \eta_j, \eta_0)$$

Case 4: $i, j, k \neq 0$. As in Case 2, all the extra terms of (3.24) vanish, leaving

$$f_{ijk} = 2\,\mathcal{Y}(\eta_i, \eta_j, \eta_k)$$

thus finally completing the proof of Theorem 3.10. $\square$

Remark 3.13. Note that Proposition 3.9 and Theorem 3.10 give more reasons to prefer the superpotential function f that we are considering, rather than $F = -\log(f)$, since it is clear from their proofs that both F_{ij} and F_{ijk} are not as simple and natural as f_{ij} and f_{ijk}. See also Remarks 3.7, 4.9, and 5.8.

4 Intermediate Jacobians in G_2-geometry

We begin with the following observation. All of the special geometric structures that exist on a G_2-manifold M are defined using both the parallel 3-form φ and its associated Hodge-dual parallel 4-form $\psi = *\varphi$. These include associative and coassociative submanifolds of M, as well as Donaldson-Thomas bundles on M. A discussion of these structures is in Section 5. For this reason, we consider the *pair* $(\varphi, *\varphi) \in H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ as representing the G_2-structure corresponding to φ. In this context we have the following result of Joyce.

Proposition 4.1 (Joyce [17]). *By Poincaré duality, $H^4(M, \mathbb{R}) \cong H^3(M, \mathbb{R})^*$, so the space $H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ has a natural symplectic structure. Define the subset $\mathcal{L}$ of $H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ by*

$$\mathcal{L} = \{([\varphi], [*\varphi]) : \varphi \textit{ is a torsion-free } \mathrm{G}_2\textit{-structure}\}$$

Then $\mathcal{L}$ is a Lagrangian submanifold *of $H^3(M, \mathbb{R}) \oplus H^4(M, \mathbb{R})$ with respect to the natural symplectic structure.*

Proof. In the proof of Theorem 3.5, we showed that

$$\left.\frac{\partial f}{\partial t}\right|_{\varphi} = \int_M \frac{\partial \varphi}{\partial t} \wedge *\varphi$$

which says that $df|_{[\varphi]} = [*\varphi]$ under the isomorphism $H^3(M,\mathbb{R})^* = H^4(M,\mathbb{R})$. Therefore $\mathcal{L}$ is locally of the form $([\varphi], df|_{[\varphi]})$, the graph of the gradient of a real function f. Hence $\mathcal{L}$ is Lagrangian. $\square$

Remark 4.2. It is clear that Theorem 3.2 can be generalized to show that the maps $([\varphi],[*_\varphi\varphi]) \mapsto [\varphi] \in H^3(M,\mathbb{R})$ and $([\varphi],[*_\varphi\varphi]) \mapsto [*_\varphi\varphi] \in H^4(M,\mathbb{R})$ are local diffeomorphisms. In fact, by projection onto the first factor, we see $\mathcal{L}$ is diffeomorphic to $\mathcal{M}$, the G_2-moduli space.

4.1 Intermediate G_2-Jacobians

We first explain the motivation for introducing (intermediate) Jacobians. For an algebraic curve C, we associate to it the Jacobian $\mathcal{J}(C) = H^{1,0}(C) \backslash H^1(C,\mathbb{C}) / H^1(C,\mathbb{Z})$, which is an Abelian variety. It determines the curve C itself. Similarly, for any complex threefold X, we define its intermediate Jacobian to be the Abelian variety

$$\mathcal{J}(X) = \left(H^{3,0}(X) + H^{2,1}(X)\right) \backslash H^3(X,\mathbb{C}) / H^3(X,\mathbb{Z}),$$

and it captures much of the algebraic structure of X, especially when X is a Calabi-Yau threefold. Now we will define a similar object, a flat torus which encodes the information of a torsion-free G_2-structure on M^7.

From now on we assume that the cohomology of M has no torsion. If not, we need to consider *gerbes*. See Section 6 for a discussion of the general case. We use the notation

$$H^k(M,S^1) = H^k(M,\mathbb{R})/H^k(M,\mathbb{Z}) \tag{4.1}$$

which is topologically a torus of dimension $b_k(M)$. Now consider the splitting

$$H^3(M,\mathbb{R}) \oplus H^4(M,\mathbb{R}) = H' \oplus H''$$

where we define

$$\begin{aligned} H' &= (1+\star_\varphi)H^3(M,\mathbb{R}) = \{(C,\star_\varphi C) : C \in H^3(M,\mathbb{R})\} \\ H'' &= (1-\star_\varphi)H^3(M,\mathbb{R}) = \{(C,-\star_\varphi C) : C \in H^3(M,\mathbb{R})\} = (H')^\perp \end{aligned}$$

where $\star_\varphi$ is the operator of Definition 2.2. Recall we are implicitly identifying a cohomology class $[C]$ with its harmonic representative. Clearly we have $H'' \cong H^3(M,\mathbb{R}) \oplus H^4(M,\mathbb{R})/H'$. Now let $\Lambda = (1-\star_\varphi)H^3(M,\mathbb{Z})$ be the image of the lattice $H^3(M,\mathbb{Z})$ under the projection onto H''.

Definition 4.3. We define the intermediate G_2-Jacobian $\mathcal{J}_\varphi$ by

$$\begin{aligned} \mathcal{J}_\varphi &= (1+\star_\varphi)H^3(M,\mathbb{R}) \backslash H^3(M,\mathbb{R}) \oplus H^4(M,\mathbb{R})/(1-\star_\varphi)H^3(M,\mathbb{Z}) \\ &= H''/\Lambda \end{aligned} \tag{4.2}$$

for each $\varphi \in \mathcal{M}$.

By projection onto the second factor, we have the isomorphism

$$\mathcal{J}_\varphi \cong H^4(M,\mathbb{R})/\star_\varphi H^3(M,\mathbb{Z}) \tag{4.3}$$

Equation (4.3) says that we can think of $\mathcal{J}_\varphi$ as a quotient of $H^4(M,\mathbb{R})$ by a varying lattice which detects the G_2-structure.

For a Calabi-Yau 3-fold X^6 or a G_2-manifold M^7, the complement of the tangent space to the intermediate Jacobian in $H^3(X,\mathbb{C})$ or $H^3(M,\mathbb{R}) \oplus H^4(M,\mathbb{R})$, respectively, is the tangent space to the local moduli space of deformations. In the Calabi-Yau case, the space $H^{3,0} \oplus H^{2,1}$ is the infinitesmal moduli space of deformations of the holomorphic volume form Ω (which corresponds to deformations of complex structures and volume scalings). In the G_2 case, this complement is the set $H' = \{(\eta, \star_\varphi \eta) : \eta \in H^3(M,\mathbb{R})\}$ which by Lemma 2.1 and Remark 4.2 is exactly the tangent space at φ to the moduli space of torsion-free G_2-structures.

We can define a similar object $\tilde{\mathcal{J}}_\varphi$ by

$$\begin{aligned}
\tilde{\mathcal{J}}_\varphi &= (1+\star_\varphi)H^4(M,\mathbb{R})\backslash H^4(M,\mathbb{R}) \oplus H^3(M,\mathbb{R})/(1-\star_\varphi)H^4(M,\mathbb{Z}) \\
&\cong H^3(M,\mathbb{R})/\star_\varphi H^4(M,\mathbb{Z})
\end{aligned} \tag{4.4}$$

The two tori $\mathcal{J}_\varphi$ and $\tilde{\mathcal{J}}_\varphi$ are dual to each other. We can use either one to define the universal intermediate G_2-Jacobian in Section 4.2. In the former case the resulting space will be locally isomorphic to $T^*\mathcal{M}$, and in the latter case to $T\mathcal{M}$. Both viewpoints will be needed, depending on whether we are considering the associative or coassociative moduli. This is discussed in Sections 5.2 and 5.3.

4.2 The Universal Intermediate G_2-Jacobian

We now consider all the intermediate G_2-Jacobians $\mathcal{J}_\varphi$ for all torsion-free G_2-structures φ in $\mathcal{M}$.

Definition 4.4. We define the *universal intermediate* G_2*-Jacobian* $\mathcal{J}$ to be the fibre bundle over $\mathcal{M}$ whose fibre over a point φ in $\mathcal{M}$ is the intermediate Jacobian $\mathcal{J}_\varphi$ associated to φ. That is,

$$\mathcal{J} = \{(\varphi, \mathcal{J}_\varphi); \varphi \in \mathcal{M}\}$$

Remark 4.5. We also call $\mathcal{J}$ the moduli space of torsion-free G_2-structures, together with *C-fields*. The elements of the fibre $\mathcal{J}_\varphi$ over each point φ in $\mathcal{M}$ are called C-fields with respect to φ in the physics literature.

By the isomorphism (4.3), we see that topologically, $\mathcal{J}$ is isomorphic to $\mathcal{M} \times H^4(M,S^1)$. Locally, $H^4(M,S^1)$ is of course isomorphic to $H^4(M,\mathbb{R})$, and therefore $\mathcal{J}$ is locally isomorphic to $\mathcal{M} \times H^4(M,\mathbb{R}) \cong T^*\mathcal{M}$, since $H^4 = (H^3)^*$ by Poincaré duality.

Theorem 4.6. *The universal intermediate Jacobian $\mathcal{J}$ admits the structure of a pseudo-Kähler manifold.*

We are going to give two proofs of Theorem 4.6, to illustrate clearly the relationships between the various structures.

Proof 1. Let $\mathcal{U}_{\varphi_\circ}$ be a flat coordinate chart in $\mathcal{M}$ with flat coordinates $x^0, \ldots, x^n$. Then the canonical coordinates $x^0, \ldots, x^n, y_0, \ldots, y_n$ on $T^*\mathcal{U}_{\varphi_\circ}$ descend to coordinates on the restriction of $\mathcal{J}$ to $\mathcal{U}_{\varphi_\circ}$, where we use the same letter y_i to denote the angle coordinates on the fibre. Let

$$\omega = dx^i \wedge dy_i$$

be the canonical symplectic form on $\mathcal{J}$ defined using the local isomorphism with $T^*\mathcal{M}$. It is easy to see that this is globally well-defined on $\mathcal{J}$.

Now we define new local coordinates $x_0, \ldots, x_n$ on the base $\mathcal{U}_{\varphi_\circ}$ by

$$x_k = \frac{\partial f}{\partial x^k} \tag{4.5}$$

where f is the superpotential function. These are the new coordinates which are used to define the Legendre transform of the function f. To see that these coordinates are well defined, observe that (4.5) implies

$$dx_k = f_{kj} dx^j \tag{4.6}$$

where $f_{kj} = \frac{\partial^2 f}{\partial x^k \partial x^j}$ is invertible. Note that the right hand side is closed, since $f_{kjl} = f_{klj}$, and thus it is exact in $\mathcal{U}_{\varphi_\circ}$ (we can assume that the flat coordinate charts are contractible.) Therefore the new coordinates x_k are well defined up to a constant, which we can fix by demanding that the origin corresponds to $\varphi_\circ$, the origin with respect to the original coordinates. Now

$$\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_0}, \ldots \frac{\partial}{\partial y_n}$$

is a local basis of coordinate vector fields on $\mathcal{J}$. We define an endomorphism J of the tangent bundle of $\mathcal{J}$ by

$$J\left(\frac{\partial}{\partial x_k}\right) = \frac{\partial}{\partial y_k} \qquad J\left(\frac{\partial}{\partial y_k}\right) = -\frac{\partial}{\partial x_k} \qquad k = 0, \ldots, n$$

It is a simple exercise to check that if $x^i = P^i_j \tilde{x}^j + Q^j$ for two overlapping flat coordinate charts, then $d\tilde{x}_k = P^i_k dx_i$ and $d\tilde{y}_k = P^i_k dy_i$, hence J is well-defined globally on $\mathcal{J}$. This was the reason for considering the coordinate transformation from x^i to x_i. This is the Legendre transform corresponding to f, as described for example in [26]. It is clear that $J^2 = -1$ and that in fact J is an *integrable* complex structure on $\mathcal{J}$, since $z_k = x_k + iy_k$ are local holomorphic coordinates. Now let us define a metric $\mathcal{G}_{\mathcal{J}}$ on $\mathcal{J}$ by the compatibility relation

$$\mathcal{G}_{\mathcal{J}}(X, Y) = \omega(X, JY) \tag{4.7}$$

It follows easily from $J^2 = -1$ and (4.7) that $\mathcal{G}_{\mathcal{J}}$ is J-invariant. From (4.6) it is clear that

$$\frac{\partial}{\partial x^k} = f_{kj} \frac{\partial}{\partial x_j}$$

Let f^{ij} be the inverse matrix of f_{ij}. With respect to the coordinates x^i, y_i, the metric $\mathcal{G}_{\mathcal{J}}$ is

$$\begin{aligned}
\mathcal{G}_{\mathcal{J}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= \omega\left(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial x^j}\right) = f_{jl}\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_l}\right) = f_{ji} \\
\mathcal{G}_{\mathcal{J}}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_j}\right) &= \omega\left(\frac{\partial}{\partial x^i}, J\frac{\partial}{\partial y_j}\right) = -f^{jl}\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^l}\right) = 0 \\
\mathcal{G}_{\mathcal{J}}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) &= \omega\left(\frac{\partial}{\partial y_i}, J\frac{\partial}{\partial y_j}\right) = -f^{jl}\omega\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial x^l}\right) = f^{ji}
\end{aligned}$$

which we can summarize as

$$\mathcal{G}_{\mathcal{J}} = \mathcal{G}_{ij} dx^i \otimes dx^j + \mathcal{G}^{ij} dy_i \otimes dy_j \tag{4.8}$$

recalling that $f_{ij} = \mathcal{G}_{ij}$. Thus $(\omega, J, \mathcal{G}_{\mathcal{J}})$ is a pseudo-Kähler structure on $\mathcal{J}$. □

Proof 2. Alternatively, we could proceed as follows. Theorem 3.5 gives us the Riemannian metric $\mathcal{G}_{ij}$ on the base $\mathcal{M}$ of $\mathcal{J}$. We use the flat connection ∇ on $\mathcal{M}$ to lift the coordinate vector fields $\eta_i = \frac{\partial}{\partial x^i}$ to their *horizontal lifts* $\tilde{\eta}_i = \frac{\partial}{\partial x^i}$ (since $\nabla = d + A$ with $A = 0$.) Now we use the connection ∇ to lift the metric $\mathcal{G}$ from $\mathcal{M}$ to a metric $\mathcal{G}_{\mathcal{J}}$ on $\mathcal{J}$ by isometrically identifying the horizontal space with the tangent space to the base, and using the fibre metric on the cotangent bundle induced from the metric $\mathcal{G}$ on $\mathcal{M}$, since $\mathcal{J}$ is locally isomorphic to $T^*\mathcal{M}$. Explicitly,

$$\begin{aligned}
\mathcal{G}_{\mathcal{J}}(\tilde{\eta}_i, \tilde{\eta}_j) &\equiv \mathcal{G}(\eta_i, \eta_j) = \mathcal{G}_{ij} \\
\mathcal{G}_{\mathcal{J}}\left(\tilde{\eta}_i, \frac{\partial}{\partial y_j}\right) &\equiv 0 \\
\mathcal{G}_{\mathcal{J}}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) &\equiv \mathcal{G}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \mathcal{G}^{ij}
\end{aligned}$$

From (4.8) we see that this is the same metric $\mathcal{G}_{\mathcal{J}}$ again. Let ω be the standard symplectic form on $\mathcal{J}$ as before. Now we can use (4.7) to define J. It is easy to check using (4.8) and (4.7) that this gives

$$J\left(\frac{\partial}{\partial x^i}\right) = \mathcal{G}_{ij}\frac{\partial}{\partial y_j} \qquad\qquad J\left(\frac{\partial}{\partial y_i}\right) = -\mathcal{G}^{ij}\frac{\partial}{\partial x^j} \tag{4.9}$$

from which it follows that $J^2 = -1$. Now $\zeta_k = \frac{\partial}{\partial x^k} - i\mathcal{G}_{kj}\frac{\partial}{\partial y_j}$ is a basis of $(1,0)$ vector fields. The Lie bracket of two $(1,0)$ vector fields of this type is easily computed to be

$$[\zeta_k, \zeta_l] = i\left(\frac{\partial \mathcal{G}_{kj}}{\partial x^l}\frac{\partial}{\partial y_j} - \frac{\partial \mathcal{G}_{lj}}{\partial x^k}\frac{\partial}{\partial y_j}\right) = 0$$

since $\mathcal{G}_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ is a Hessian. Therefore the $[1,0]$ vector fields are closed under the Lie bracket, and thus by the Newlander-Nirenberg theorem, J is integrable. Hence $(\omega, J, \mathcal{G}_{\mathcal{J}})$ is a pseudo-Kähler structure on $\mathcal{J}$. □

We pause now to describe $g_{\mathcal{J}}$, ω, and J invariantly. At a point (φ, D) in $\mathcal{M} \times H^4(M, S^1) \cong \mathcal{J}$, the canonical symplectic form ω on $\mathcal{J}$ is given by

$$\omega((\eta_1, \theta_1), (\eta_2, \theta_2)) = \int_M \eta_1 \wedge \theta_2 - \int_M \eta_2 \wedge \theta_1 \tag{4.10}$$

where $\eta_i \in T_\varphi \mathcal{M} = H^3(M, \mathbb{R})$ and $\theta_i \in T_D(H^4(M, S^1)) \cong H^4(M, \mathbb{R}) = (H^3(M, \mathbb{R}))^*$, by Poincaré duality. Also, using (2.2), (3.3), and (4.8) we see

$$\mathcal{G}_{\mathcal{J}}((\eta_1, \theta_1), (\eta_2, \theta_2)) = \int_M \eta_1 \wedge \star_\varphi \eta_2 + \int_M \theta_1 \wedge \star_\varphi \theta_2 \tag{4.11}$$

Now from (4.7), (4.10), and (4.11) it follows that

$$J(\eta, \theta) = (-\star_\varphi \theta, \star_\varphi \eta) \tag{4.12}$$

Since $\star_\varphi^2 = 1$, this equation shows clearly that $J^2 = -1$.

Corollary 4.7. *The fibration $\pi : \mathcal{J} \to \mathcal{M}$ is a Lagrangian fibration with a Lagrangian section. That is, the zero section $\mathcal{M}$ and the fibres of the projection map π are Lagrangian submanifolds of $\mathcal{J}$.*

Proof. This is immediate from the fact that ω is the canonical symplectic form on $\mathcal{J}$ given by the local isomorphism with the cotangent bundle $T^*\mathcal{M}$. $\square$

We also have the following further relationship between the superpotential function f and the pseudo-Kähler structure on $\mathcal{J}$.

Proposition 4.8. *The Legendre transform $\hat{f}$ of f is a Kähler potential for the pseudo-Kähler structure on $\mathcal{J}$. That is,*

$$\omega = 2i\partial\bar{\partial}\hat{f} = 2i\left(\frac{\partial^2 \hat{f}}{\partial z_i \partial \bar{z}_j}\right) dz_i \wedge d\bar{z}_j$$

Furthermore, $\hat{f}(\varphi) = \frac{4}{3}f(\varphi) = \frac{4}{7}\int_M(\varphi \wedge *_\varphi \varphi)$.

Proof. The definition of the Legendre transform $\hat{f}$ of f is

$$\hat{f}(x_0, \ldots, x_n) = x_i x^i(x_0, \ldots, x_n) - f(x^0(x_0, \ldots, x_n), \ldots, x^n(x^0, \ldots, x_n))$$

as a function of the x_i's, where recall that $x_i = \frac{\partial f}{\partial x^i}$. An easy calculation shows

$$\frac{\partial^2 \hat{f}}{\partial x_i \partial x_j} = f^{ij}$$

where f^{ij} is the inverse matrix of $f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$. Since ω is J-invariant, it is of type $(1,1)$, and we have

$$\omega = \frac{i}{2}\mathcal{G}^{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta = \frac{i}{2}\frac{\partial^2 \hat{f}}{\partial x_\alpha \partial x_\beta} dz_\alpha \wedge d\bar{z}_\beta = 2i\frac{\partial^2 \hat{f}}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta$$

and hence $\omega = 2i\partial\bar{\partial}\hat{f}$. All that remains is to establish that $\hat{f} = \frac{4}{3}f$. Let $\varphi = x^i \eta_i$. Using the computations in the proof of Theorem 3.5, we see that

$$\begin{aligned}
\hat{f} &= x^i x_i - f = x^i \frac{\partial f}{\partial x^i} - f = x^i \int_M \eta_i \wedge *_\varphi \varphi - \frac{3}{7}\int_M \varphi \wedge *_\varphi \varphi \\
&= \int_M \varphi \wedge *_\varphi \varphi - \frac{3}{7}\int_M \varphi \wedge *_\varphi \varphi = \frac{4}{7}\int_M \varphi \wedge *_\varphi \varphi
\end{aligned}$$

as claimed. $\square$

Remark 4.9. Using $F = -\log(f)$ as a superpotential, one can still obtain a Kähler structure on $\mathcal{J}$. In fact this time it is really Kähler since the Hessian metric F_{ij} on $\mathcal{M}$ is Riemannian (if and only if M is irreducible.) However, the Legendre transform $\hat{F}$ is *not* a constant multiple of F in this case. See also Remarks 3.7, 3.13, and 5.8.

The pseudo-Kähler structure on $\mathcal{J}$ we have just described arises from the pseudo-Riemannian metric $\mathcal{G}$ on $\mathcal{M}$ together with the local isomorphism of $\mathcal{J}$ with $T^*\mathcal{M}$. This structure will be used to study the moduli space of branes (coassociative submanifolds) of M in Section 5.3.

Alternatively, we could consider the other definition of the intermediate Jacobian $\tilde{\mathcal{J}}_\varphi$ given in (4.4), which gives us that the universal intermediate Jacobian $\mathcal{J}$ is locally isomorphic to $T\mathcal{M}$.

We take $x^0, \dots, x^n$ to be local flat coordinates on a flat chart $\mathcal{U}_{\varphi_\circ}$ as before, and let $y^0, \dots, y^n$ be the corresponding coordinates on the fibre. This gives a local basis of coordinate vector fields

$$\frac{\partial}{\partial x^0}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^0}, \dots \frac{\partial}{\partial y^n}$$

and we define an endomorphism $\tilde{J}$ of the tangent bundle of $\mathcal{J}$ by

$$\tilde{J}\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k} \qquad \tilde{J}\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k} \qquad k = 0, \dots, n$$

Then $\tilde{J}$ is an integrable complex structure on $\mathcal{J}$ with local holomorphic coordinates $z^k = x^k + iy^k$.

As before we use the flat connection ∇ on $\mathcal{M}$ to lift the metric $\mathcal{G}$ on $\mathcal{M}$ to a metric $\tilde{\mathcal{G}_{\mathcal{J}}}$ on $\mathcal{J}$ by isometrically identifying the horizontal space with the tangent space to the base, and using the fibre metric on the tangent bundle induced from the metric $\mathcal{G}$ on $\mathcal{M}$, since $\mathcal{J}$ is locally isomorphic to $T\mathcal{M}$. That is,

$$\tilde{\mathcal{G}_{\mathcal{J}}} = \mathcal{G}_{ij} dx^i \otimes dx^j + \mathcal{G}_{ij} dy^i \otimes dy^j$$

It is clear that this metric $\tilde{\mathcal{G}_{\mathcal{J}}}$ is $\tilde{J}$-invariant. This time we use the compatibility relation to define the symplectic form $\tilde{\omega}$ from $\tilde{J}$ and $\tilde{\mathcal{G}_{\mathcal{J}}}$ by

$$\tilde{\omega}(X, Y) = \tilde{\mathcal{G}_{\mathcal{J}}}(\tilde{J}X, Y)$$

In coordinates, this becomes

$$\tilde{\omega} = \mathcal{G}_{ij} dx^i \wedge dy^j$$

which is closed because $\frac{\partial \mathcal{G}_{ij}}{\partial x^l} = \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^l}$ is symmetric in i and l. Equivalently we can write $\tilde{\omega} = dx_j \wedge dy^j$ where $dx_j = \mathcal{G}_{ij} dx^i$ as before, which clearly shows the closure of $\tilde{\omega}$.

The triple $(\tilde{\omega}, \tilde{J}, \tilde{\mathcal{G}_{\mathcal{J}}})$ is a different pseudo-Kähler structure on $\mathcal{J}$. In this case it is easy to check directly that the superpotential function f is exactly a Kähler potential for this structure:

$$\tilde{\omega} = 2i\partial\bar{\partial} f = 2i \left(\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j}\right) dz^i \wedge d\bar{z}^j$$

This pseudo-Kähler structure is used in Section 5.2 to study the moduli space of instantons (associative submanifolds) of M. The two structures are clearly isomorphic, and it is easy to see that they are related by the Legendre transform. See [26] for a similar construction in the context of mirror symmetry of semi-flat Calabi-Yau manifolds.

We will also need the invariant descriptions of $\tilde{\mathcal{G}_{\mathcal{J}}}$, $\tilde{\omega}$, and $\tilde{J}$. At a point (φ, C) in $\mathcal{M} \times H^3(M, S^1) \cong \mathcal{J}$, let $\eta_i \in T_\varphi \mathcal{M} = H^3(M, \mathbb{R})$ and $\mu_i \in T_C(H^3(M, S^1)) \cong H^3(M, \mathbb{R})$. It is easy to check that

$$\tilde{\omega}((\eta_1, \mu_1), (\eta_2, \mu_2)) = \int_M \eta_1 \wedge \star_\varphi \mu_2 - \int_M \eta_2 \wedge \star_\varphi \mu_1 \tag{4.13}$$

and

$$\tilde{\mathcal{G}_{\mathcal{J}}}((\eta_1, \mu_1), (\eta_2, \mu_2)) = \int_M \eta_1 \wedge \star_\varphi \eta_2 + \int_M \mu_1 \wedge \star_\varphi \mu_2 \tag{4.14}$$

and

$$\tilde{J}(\eta, \mu) = (-\mu, \eta) \tag{4.15}$$

These should be compared with (4.10), (4.11), and (4.12), respectively.

We close this section with a discussion of the cubic form associated to a Lagrangian fibration, and show that in this case it is precisely the Yukawa coupling from Definition 3.8. From the work of Donagi and Markman [8], a Lagrangian fibration with section corresponds to a certain cubic form on the base of the fibration.

The construction of the cubic form associated to the Lagrangian fibration is as follows. By Remark 4.2, we can identify the tangent space $T_\varphi\mathcal{M}$ to the base $\mathcal{M}$ with $H^4(M,\mathbb{R})$. For any tangent vector $\theta \in H^4(M,\mathbb{R})$ to the base, we get an infinitesimal variation of the Lagrangian torus fibers $H^4(M,S^1)$. Note that here we are fixing the lattice and changing the metric on $H^4(M,\mathbb{R})$, instead of fixing the metric on $H^4(M,\mathbb{R})$ and varying the lattice. Such an infinitesmal variation is an element of $\mathrm{Sym}^2(T^*H^4(M,\mathbb{R})) \cong \mathrm{Sym}^2(H^3(M,\mathbb{R}))$. We therefore have a map

$$c : T\mathcal{M} \to \mathrm{Sym}^2(H^3(M,\mathbb{R}))$$

or equivalently an element c of

$$T^*\mathcal{M} \otimes \mathrm{Sym}^2(H^3(M,\mathbb{R})) \cong H^3(M,\mathbb{R}) \otimes \mathrm{Sym}(H^3(M,\mathbb{R}))$$

We will show that the Lagrangian condition in fact implies that

$$c \in \mathrm{Sym}^3(H^3(M,\mathbb{R}))$$

by showing that $c = 2\,\mathcal{Y}$, where $\mathcal{Y}$ is the Yukawa coupling. By the definition of c, if we take η_1, η_2, and η_3 to be elements of $H^3(M,\mathbb{R}) \cong T_\varphi\mathcal{M}$, then

$$c\,(\eta_1,\eta_2,\eta_3) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{G}_t\,(\eta_1,\eta_2) = \left.\frac{d}{dt}\right|_{t=0} \int_M \eta_1 \wedge \star_{\varphi_t}\eta_2$$

where $\mathcal{G}_t$ is the Hessian metric of Theorem 3.5 at $T_{\varphi_t}\mathcal{M}$, for φ_t satisfying $\varphi_0 = \varphi$ and $\left.\frac{d}{dt}\right|_{t=0}\varphi_t = \eta_3$. But the claim now follows immediately from Theorem 3.10.

5 Abel-Jacobi Maps in G_2-geometry

In this section, we discuss Abel-Jacobi type maps in G_2-geometry. We show that associative cycles, coassociative cycles, and deformed Donaldson-Thomas connections are critical points of Chern-Simons type functionals. The moduli spaces of these structures can be isotropically immersed (with respect to the appropriate symplectic structure) into the universal intermediate Jacobian $\mathcal{J}$ using these Abel-Jacobi maps.

5.1 Functionals of Chern-Simons Type

We are going to define a Chern-Simons type functional for pairs (N,A) where N is a k-dimensional submanifold of M, of a fixed diffeomorphism type, and A is a unitary connection for a fixed rank r complex vector bundle E over N. Let $\mathcal{C}_k$ be the set of all such pairs. We will fix a base point (N_0,A_0) on each connected component of the configuration space $\mathcal{C}_k$. For simplicity, we will restrict our attention to one such connected component and denote its univeral cover by $\widetilde{\mathcal{C}_k}$. An element in

$\widetilde{\mathcal{C}_k}$ corresponds to a pair (N, A) in $\mathcal{C}_k$ together with a homotopy class of paths in $\widetilde{\mathcal{C}_k}$ joining it with (N_0, A_0). Let $\{(N_t, A_t)\}_{t\in[0,1]}$ be any such path. We define the set

$$\bar{N} = \{(t, p_t) : t \in [0,1], p_t \in N_t\} \tag{5.1}$$

Clearly we have $\bar{N} \cong [0,1] \times N_0$ since each N_t is diffeomorphic to N_0. Let $\pi : \bar{N} \to M$ be the projection onto the second factor: $\pi(t, p_t) = p_t \in N_t \subset M$. Then $\pi^*(\varphi)$ and $\pi^*(*\varphi)$ are forms on $\bar{N}$. From the family of connections A_t on N_t we define

$$\bar{A}(t, p_t) = \pi^*(A_t(p_t))$$

which is unitary connection $\bar{A}$ on $\bar{N}$.

Definition 5.1. We define the Chern-Simons type functional $\Phi_{k,\varphi}$ for fixed k and fixed torsion-free G_2-structure φ as follows:

$$\begin{aligned} \Phi_{k,\varphi} &: \widetilde{\mathcal{C}_k} \to \mathbb{R} \\ \Phi_{k,\varphi}(N, A) &= \int_{\bar{N}} \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi} F_{\bar{A}} + \pi^*(\varphi) + \pi^*(*\varphi)\right)\right] \end{aligned} \tag{5.2}$$

Here $F_{\bar{A}}$ is locally a matrix-valued differential form, and we consider the ordinary forms φ and $*\varphi$ to be matrix-valued by tensoring with the identity matrix. Hence $\mathrm{Tr}(\alpha) = r\alpha$ for an ordinary form α, where r is the rank of E.

By standard arguments using the closedness of φ and $*\varphi$ and the Bianchi identity, this integral is independent of the choice of the path in $\widetilde{\mathcal{C}_k}$. Therefore, $\Phi_{k,\varphi}$ is a well-defined function. We will sometimes abuse notation and drop the π^* for notational simplicity, when there is no risk of confusion.

Remark 5.2. If $\bar{N}$ is a *closed* manifold, then Stokes' theorem shows that $\Phi_{k,\varphi}$ is a well-defined functional on $\mathcal{C}_k$, modulo discrete periods depending on the cohomology classes $[F_A]$, $[\varphi]$, and $[*\varphi]$. Therefore, up to a constant multiple, we can assume that $\Phi_{k,\varphi}(N, A)$ is always an integer for a closed manifold $\bar{N}$, provided that $[\varphi]$ and $[*\varphi]$ are rational cohomology classes. In such case, we can descend $\Phi_{k,\varphi}$ to a *circle-valued function* on $\mathcal{C}_k$.

We will need the explicit form of this functional for paths such that $N_t = N_0$ for all t. In this case $\bar{N} = [0,1] \times N_0$. Then A_t is a connection on N_0 for each t, and t is just a parameter. We have $F_{\bar{A}} = d\bar{A} + \bar{A} \wedge \bar{A}$, where d is the exterior derivative on $[0,1] \times N_0$. Hence $F_{\bar{A}} = d_{N_0} A_t + dt \wedge A_t' + A_t \wedge A_t = F_{A_t} + dt \wedge A_t'$, where $'$ denotes differentiation with respect to the parameter t. Then it is easy to see that

$$\exp\left(\frac{i}{2\pi} F_{\bar{A}} + \varphi + *\varphi\right) = \left(1 + \frac{i}{2\pi} dt \wedge A_t'\right) \wedge \exp\left(\frac{i}{2\pi} F_{A_t} + \varphi + *\varphi\right)$$

The first term above has no dt factor, so it integrates to zero over $\bar{N}$. Therefore we are left with

$$\Phi_{k,\varphi}(N_0, A) = \frac{i}{2\pi} \int_0^1 \int_{N_0} dt \wedge \mathrm{Tr}\left(A_t' \wedge \exp\left(\frac{i}{2\pi} F_{A_t} + \varphi + *\varphi\right)\right) \tag{5.3}$$

whenever A_t is a path of connections from A_0 to A, for fixed N_0.

The symmetry group for this functional $\Phi_{k,\varphi}$ is the connected component of the group of extended gauge symmetries $\widetilde{\mathcal{G}}$ of the bundle E over N_0, which fits into the following exact sequence:

$$1 \to \mathcal{G} \to \widetilde{\mathcal{G}} \to \mathrm{Diff}_0(N_0) \to 1$$

where $\mathcal{G} = \mathrm{Aut}(E)$ is the group of unitary gauge transformations of E, and $\mathrm{Diff}_0(N_0)$ is the group of diffeomorphisms of N_0 isotopic to the identity.

We denote the moduli space $\mathcal{N}_{k,\varphi}$ of critical points of $\Phi_{k,\varphi}$ by

$$\mathcal{N}_{k,\varphi} = \{d\Phi_{k,\varphi} = 0\} / \widetilde{\mathcal{G}} \tag{5.4}$$

and the associated *universal moduli space* by

$$\mathcal{N}_k = \coprod_{\varphi \in \mathcal{M}} \mathcal{N}_{k,\varphi} \tag{5.5}$$

Because we have $\exp(\varphi + *\varphi) = 1 + \varphi + *\varphi + \varphi \wedge *\varphi$, we will see that the functional Φ_k is only interesting for $k = 3$, 4, or 7. We are going to see that the corresponding critical points are associative cycles, coassociative cycles and deformed Donaldson-Thomas connections.

We now proceed to generalize the functionals $\Phi_{k,\varphi}$ to obtain analogues of *Abel-Jacobi maps* for G_2-manifolds. Note that since we are assuming that the cohomology of M has no torsion, we have

$$(H^k(M, S^1))^* = H^{7-k}(M, S^1)$$

where recall that $H^k(M, S^1) = H^k(M, \mathbb{R})/H^k(M, \mathbb{Z})$. Let $l = 3$ or 4. We have shown that $\mathcal{J} \cong \mathcal{M} \times H^3(M, S^1)$ and also $\mathcal{J} \cong \mathcal{M} \times H^4(M, S^1)$.

Definition 5.3. We define the Abel-Jacobi map ν_k^l from the universal moduli space $\mathcal{N}_k$ to $\mathcal{J} \cong \mathcal{M} \times H^{7-l}(M, S^1) \cong \mathcal{M} \times (H^l(M, S^1))^*$ as follows.

$$\begin{array}{rcccl} \nu_k^l & : & \mathcal{N}_k & \to & \mathcal{M} \times (H^l(M, S^1))^* \\ & & (\varphi, N, A) & \mapsto & (\varphi, \beta) \end{array} \tag{5.6}$$

where $\beta \in (H^l(M, S^1))^*$ is defined by

$$\int_M \beta \wedge \alpha = \int_{\bar{N}} \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi} F_{\bar{A}} + \pi^*(\varphi) + \pi^*(*\varphi)\right)\right] \wedge \pi^*(\alpha) \tag{5.7}$$

for all $\alpha \in H^l(M, S^1)$.

In the following sections, we will examine these maps in three situations. For $k = 3$ or 7 we need $l = 4$, and for $k = 4$ we need $l = 3$. We will see explicitly that the Abel-Jacobi map ν_k^l is well-defined in each of these cases, and we will show that its image is isotropic with respect to the appropriate symplectic structure. Hence $\mathcal{N}_k$ is a Lagrangian submanifold of $\mathcal{J}$ whenever ν is an immersion.

We close this section with the following lemma, which we will need later.

Lemma 5.4. *Let (M^n, g) be a compact oriented Riemannian manifold, with N_0 a closed oriented k-dimensional submanifold of M. Suppose that X is a normal vector field on N_0. That is, $X(p) \perp T_p N_0$ for all $p \in N_0$. Define*

$$N_s = \exp(sX) \cdot N_0 \qquad , \qquad \bar{N}(t) = \{(t, p_t) : t \in [0,1], p_t \in N_t\}$$

and define $\pi : \bar{N}(t) \to M$ to be the projection onto the second factor. We give $\bar{N}(t)$ the orientation such that $(\frac{\partial}{\partial t}, e_1, \ldots, e_k)$ is an oriented basis of $\bar{N}(t)$ whenever $(e_1, \ldots, e_k)$ is an oriented basis of N_s. If α is a $(k+1)$-form on M, then

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\bar{N}(t)} \pi^*(\alpha) = \int_{N_0} X \lrcorner\, \alpha \tag{5.8}$$

Proof. Define a map $\rho : [0,t] \times N_0 \to \bar{N}(t)$ by $\rho(s,p) = (s, \exp(sX)\cdot p)$. Clearly ρ is a diffeomorphism, and $(\pi \circ \rho)_*(\frac{\partial}{\partial s}) = X$. We have

$$\begin{aligned}\int_{\bar{N}(t)} \pi^*(\alpha) &= \int_{[0,t]\times N_0} \rho^*(\pi^*(\alpha)) = \int_{[0,t]\times N_0} ds \wedge \left(\frac{\partial}{\partial s} \lrcorner\, (\pi\circ\rho)^*\alpha\right) \\ &= \int_0^t \left(\int_{N_0} \frac{\partial}{\partial s} \lrcorner\, (\pi\circ\rho)^*\alpha\right) ds = \int_0^t \left(\int_{N_0} X \lrcorner\, \alpha\right) ds\end{aligned}$$

and thus by the fundamental theorem of calculus,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \int_{\bar{N}(t)} \pi^*(\alpha) = \int_{N_0} X \lrcorner\, \alpha$$

which completes the proof. □

5.2 Universal moduli of instantons

An *instanton* in the G_2-manifold M^7 is an *associative* submanifold N^3. They are 3-dimensional submanifolds of M which are calibrated with respect to φ. A 3-manifold N^3 in M is associative if and only if

$$(X \lrcorner\, \psi)|_N = 0 \tag{5.9}$$

for all normal vector fields X on N, where $\psi = *\varphi$ is the dual 4-form. This is equivalent to the vanishing on N of the vector valued 3-form χ obtained by raising an index of ψ with the metric g_φ.

We will denote the functional $\Phi_{k,\varphi}$ defined in (5.2) by $\Phi_{\mathcal{A}}$ when $k = 3$. Explictly,

$$\Phi_{\mathcal{A}}(N, A) = \Phi_{3,\varphi}(N, A) = \int_{\bar{N}} \left(r\, \pi^*(\psi) - \frac{1}{8\pi^2}\,\mathrm{Tr}(F_{\bar{A}}^2)\right) \tag{5.10}$$

in this case. The critical points of $\Phi_{\mathcal{A}}$ are described by the following theorem.

Theorem 5.5. *The pair (N_0, A_0) is a critical point of $\Phi_{\mathcal{A}}$ if and only if N_0 is an associative submanifold of M and A_0 is a* flat *connection.*

Proof. We will choose (N_0, A_0) to be our base point. Consider first a variation of the connection: $A_t = A_0 + tA$, where $N = N_0$ is fixed. Hence $F_{A_t} = dA_t + A_t \wedge A_t = F_{A_0} + t(dA + A \wedge A_0 + A_0 \wedge A) + t^2(A \wedge A)$.

Then using equation (5.3), (where $A'_t = A$) we have

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_{\mathcal{A}}(N_0, A_t) &= \left.\frac{\partial}{\partial t}\right|_{t=0} - \frac{1}{4\pi^2}\int_0^t \int_{N_0} ds \wedge \mathrm{Tr}\,(A \wedge F_{A_s}) \\ &= -\frac{1}{4\pi^2}\int_{N_0} \mathrm{Tr}(A \wedge F_{A_0})\end{aligned}$$

This must vanish for all possible A, thus $F_{A_0} = 0$ and A_0 is a flat connection.

Now let N_t be a variation of N_0, which we can write as $N_t = \exp(tX) \cdot N_0$, for some normal vector field X on N_0. By Lemma 5.4, we have

$$\frac{\partial}{\partial t}\bigg|_{t=0} \Phi_{\mathcal{A}}(N_t, A_0) = r \int_{N_0} (X \lrcorner \psi) - \frac{1}{8\pi^2} \int_{N_0} \left(X \lrcorner \operatorname{Tr}(F_{A_0}^2) \right)$$

The second term vanishes since we just showed for (N_0, A_0) a critical point of $\Phi_{\mathcal{A}}$, we must have $F_{A_0} = 0$. It follows from (5.9) that (N_0, A_0) is critical for all variations if and only if N_0 is associative and A_0 is flat. $\square$

Definition 5.6. The pair (N, A) where N is an associative submanifold of M with respect to φ and A is a flat connection on N will be called an *associative cycle*. Using the notation of (5.4) we see that $\mathcal{N}_{3,\varphi}$ is the space of associative cycles with respect to the G_2-structure φ.

Now consider the Abel-Jacobi map from Definition (5.3) with $k = 3$. It is easy to see that this map is non-trivial only for $l = 4$. Let us denote ν_3^4 by ν. Explictly, we have

$$\nu(\varphi, N, A) = (\varphi, \bar{\nu}(N, A)) = (\varphi, C)$$

where $C = \bar{\nu}(N, A) \in H^3(M, S^1)$ is defined by

$$\int_M \bar{\nu}(N, A) \wedge D = r \int_{\bar{N}} \pi^*(D) \tag{5.11}$$

for any $D \in H^4(M, S^1)$, where r is the rank of E.

Proposition 5.7. *The image of ν in $\mathcal{J}$ is isotropic with respect to the symplectic structure $\tilde{\omega}$ on $\mathcal{J} \cong \mathcal{M} \times H^3(M, S^1)$ described in* (4.13).

Proof. The symplectic form is exact: $\tilde{\omega} = d\tilde{\alpha}$, where $\tilde{\alpha}$ is the 1-form given by

$$\tilde{\alpha}_{(\varphi, C)}(\eta, \mu) = \frac{1}{2} \int_M \varphi \wedge \star_\varphi \mu - \frac{1}{2} \int_M C \wedge \star_\varphi \eta \tag{5.12}$$

for $\eta \in T_\varphi \mathcal{M} \cong H^3(M, \mathbb{R})$ and $\mu \in T_C H^3(M, S^1) \cong H^3(M, \mathbb{R})$. (This follows easily from the local coordinate expression: $\tilde{\omega} = dx_j \wedge dy^j = d(\frac{1}{2} x_j dy^j - \frac{1}{2} y^j dx_j)$. We compute the pullback $\nu^*(\tilde{\alpha})$ of $\tilde{\alpha}$ by ν:

$$\begin{aligned}
&\nu^*(\tilde{\alpha})|_{(\varphi, N, A)} (\delta\varphi, \delta N, \delta A) = \tilde{\alpha}|_{(\varphi, \bar{\nu}(N, A))} (\delta\varphi, \bar{\nu}_*(\delta N, \delta A)) \\
&= \frac{1}{2} \int_M \varphi \wedge \star_\varphi \bar{\nu}_*(\delta N, \delta A) - \frac{1}{2} \int_M \bar{\nu}(N, A) \wedge \star_\varphi \delta\varphi \\
&= \frac{2}{3} \int_M \bar{\nu}_*(\delta N, \delta A) \wedge \psi - \frac{1}{2} \int_M \bar{\nu}(N, A) \wedge \star_\varphi \delta\varphi
\end{aligned} \tag{5.13}$$

where we have used the fact that $\star_\varphi \varphi = \frac{4}{3} *_\varphi \varphi = \frac{4}{3} \psi$. If we differentiate equation (5.11) with $D = \psi = *_\varphi \varphi$, and use Lemma 5.4, we obtain

$$\int_M \bar{\nu}_*(\delta N, \delta A) \wedge \psi + \int_M \bar{\nu}(N, A) \wedge \star_\varphi \delta\varphi = r \int_N (X \lrcorner \psi) + r \int_{\bar{N}} \pi^*(\star_\varphi \delta\varphi)$$

where X is the normal vector field on N corresponding to the variation δN, and we have also used Lemma 2.1. Using (5.11) again, the above expression reduces to

$$\int_M \bar{\nu}_*(\delta N, \delta A) \wedge \psi = r \int_N (X \lrcorner \psi)$$

Substituting this into (5.13) and using (5.11) again gives

$$\nu^*(\tilde{\alpha})|_{(\varphi,N,A)} (\delta\varphi, \delta N, \delta A) = \frac{2r}{3} \int_N (X \lrcorner \psi) - \frac{r}{2} \int_{\bar{N}} \pi^*(\star_\varphi \delta\varphi)$$

The first term vanishes since N is associative. Now consider again the functional $\Phi_{3,\varphi}$, but consider also variations in φ. We see that

$$d\Phi_{3,\varphi}|_{(\varphi,N,A)} (\delta\varphi, \delta N, \delta A) = r \int_{\bar{N}} \pi^*(\star_\varphi \delta\varphi)$$

since the variations in N and A vanish because (N, A) is an associative cycle, and we have used Lemma 2.1 to compute the variation in φ. Comparing the two expressions, we see

$$\nu^*(\tilde{\alpha}) = -\frac{1}{2} d\Phi_{3,\varphi}$$

and hence $\nu^*(\tilde{\omega}) = \nu^*(d\tilde{\alpha}) = d\nu^*(\tilde{\alpha}) = -\frac{1}{2} d^2\Phi_{3,\varphi} = 0$ as claimed. □

Remark 5.8. The proof of Proposition 5.7 fails if one instead uses the function $F = -\log(f)$ as the superpotential. This should be compared to Remarks 3.7, 3.13, and 4.9.

An interesting question concerns the non-degeneracy of the critical points of $\Phi_{3,\varphi}$. In [31] McLean proved that the local deformations of associative submanifolds can be (and often are) obstructed. It seems reasonable that our functional is non-degenerate at a critical point (N_0, A_0) if and only if the deformations are unobstructed at that point. The dimension of the image $\nu(\mathcal{N}_3)$ of the Abel-Jacobi map should also depend on the degeneracy of the critical points. One would like to know exactly when this image is half-dimensional, so the image is an honest Lagrangian submanifold.

The expected dimension of the moduli space of associative submanifolds on a given G_2-manifold is zero, since it is given by the index of a twisted Dirac operator on an odd-dimensional manifold. If the kernel of this Dirac operator is indeed trivial, then the moduli space is indeed a discrete set. Furthermore, almost G_2-manifolds also have a discrete moduli space of associative submanifolds. When such generic situation occurs, the image of our Abel-Jacobi map is half dimensional and therefore a Lagrangian subspace in the universal intermediate Jacobian. When the kernel is nontrivial, the moduli space of asociative submanifolds could have positive dimension. However, at the same time, such solutions may not be able to survive under deformations of G_2-structures, since the obstruction space is related to the cokernel which is also nontrivial. Our Proposition 5.7 can be interpreted as a quantitative measure of such obstructions. In fact, the image of ν may indeed be a Lagrangian subspace in general. This situation should also be compared with the case of holomorphic curves, which are the analogues of instantons for Calabi-Yau 3-folds.

5.3 Universal moduli of branes

A *brane* in the G_2-manifold M^7 is a *coassociative* submanifold L^4. They are 4-dimensional submanifolds of M which are calibrated with respect to ψ. A 4-manifold L^4 in M is coassociative if and

only if

$$\varphi|_L = 0 \tag{5.14}$$

from which it is clear they are analogues of *Lagrangian* submanifolds (see [30].)

We will denote the functional $\Phi_{k,\varphi}$ defined in (5.2) by $\Phi_{\mathcal{C}}$ when $k = 4$. Explictly,

$$\Phi_{\mathcal{C}}(L, A) = \Phi_{4,\varphi}(L, A) = \frac{i}{\pi} \int_{\bar{L}} \operatorname{Tr}\left(F_{\bar{A}} \wedge \pi^*(\varphi)\right) \tag{5.15}$$

in this case. The critical points of $\Phi_{\mathcal{C}}$ are described by the following theorem.

Theorem 5.9. *The pair (L_0, A_0) is a critical point of $\Phi_{\mathcal{C}}$ if and only if L_0 is a coassociative submanifold of M and* $\operatorname{Tr}(F_{A_0})$ *is a* self-dual 2*-form on L_0. In particular, when E is a complex line bundle (rank $r = 1$), then F_{A_0} is self-dual, so A_0 is a self-dual connection.*

Proof. We choose (L_0, A_0) to be our base point. Consider first a variation of the connection: $A_t = A_0 + tA$, where $L = L_0$ is fixed. Then using equation (5.3), (where $A'_t = A$) we have

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_{\mathcal{C}}(L_0, A_t) &= \left.\frac{\partial}{\partial t}\right|_{t=0} \frac{i}{2\pi} \int_0^t \int_{L_0} ds \wedge \operatorname{Tr}\left(A \wedge \varphi\right) \\ &= \frac{i}{2\pi} \int_{L_0} \operatorname{Tr}(A) \wedge \varphi \end{aligned}$$

This must vanish for all possible A, hence $\varphi|_{L_0} = 0$ and thus by (5.14), L_0 is coassociative.

Now let L_t be a variation of L_0, which we can write as $L_t = \exp(tX) \cdot L_0$, for some normal vector field X on L_0. By Lemma 5.4, we have

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \Phi_{\mathcal{C}}(L_t, A_0) = \frac{i}{\pi} \int_{L_0} (X \lrcorner \operatorname{Tr}(F_{A_0})) \wedge \varphi + \int_{L_0} \operatorname{Tr}(F_{A_0}) \wedge (X \lrcorner \varphi)$$

The first term vanishes since we just showed for (L_0, A_0) a critical point of $\Phi_{\mathcal{C}}$, we must have $\varphi = 0$ on L_0. When L_0 is a coassociative submanifold, the map $X \mapsto (X \lrcorner \varphi)$ gives an isomorphism between the normal vector fields X on L_0 with the *anti-self-dual* 2-forms on L_0. (See [19, 31] for details.) Therefore since the above expression must vanish for all normal vector fields X, we must have that $\operatorname{Tr}(F_{A_0})$ is orthogonal to $\Omega^2_-(L_0)$, so it is in $\Omega^2_+(L_0)$. That is, (L_0, A_0) is critical for all variations if and only if L_0 is coassociative and $\operatorname{Tr}(F_{A_0})$ is a self-dual 2-form. □

Remark 5.10. There are two different conventions for orientations in G_2-geometry. With the other convention, $(X \lrcorner \varphi)$ would be self-dual for all normal vector fields X on a coassociative, and $\operatorname{Tr}(F_{A_0})$ would be anti-self-dual in the above theorem. See [19] for more about signs and orientations in G_2-geometry.

Definition 5.11. The pair (L, A) where L is a coassociative submanifold of M with respect to φ and A is a connection on N such that $\operatorname{Tr}(F_A)$ is self-dual will be called a *coassociative cycle*. Using the notation of (5.4) we see that $\mathcal{N}_{4,\varphi}$ is the space of coassociative cycles with respect to the G_2-structure φ.

Now consider the Abel-Jacobi map from Definition (5.3) with $k = 4$. It is easy to see that this map is non-trivial only for $l = 3$. Let us denote ν_4^3 by μ. Explictly, we have

$$\mu(\varphi, L, A) = (\varphi, \bar{\mu}(L, A)) = (\varphi, D)$$

where $D = \bar{\mu}(L, A) \in H^4(M, S^1)$ is defined by

$$\int_M \bar{\mu}(L, A) \wedge C = \frac{i}{2\pi} \int_{\bar{L}} \mathrm{Tr}(F_{\bar{A}}) \wedge C \tag{5.16}$$

for any $C \in H^3(M, S^1)$.

Proposition 5.12. *The image of μ in $\mathcal{J}$ is isotropic with respect to the symplectic structure ω on $\mathcal{J} \cong \mathcal{M} \times H^4(M, S^1)$ described in* (4.10).

Proof. Since the proof is very similar to Proposition 5.7, we will be brief. This time $\omega = d\alpha$, where

$$\alpha_{(\varphi,D)}(\eta, \theta) = \frac{1}{2} \int_M \varphi \wedge \theta - \frac{1}{2} \int_M D \wedge \eta \tag{5.17}$$

for $\eta \in T_\varphi \mathcal{M} \cong H^3(M, \mathbb{R})$ and $\theta \in T_D H^4(M, S^1) \cong H^4(M, \mathbb{R})$. Now one can compute as in the proof of Proposition 5.7 that

$$\begin{aligned} \mu^*(\alpha)|_{(\varphi,L,A)} (\delta\varphi, \delta L, \delta A) = \frac{i}{4\pi} \int_L (X \lrcorner \mathrm{Tr}(F_A)) \wedge \varphi \\ + \frac{i}{4\pi} \int_L \mathrm{Tr}(F_A) \wedge (X \lrcorner \varphi) - \frac{i}{4\pi} \int_{\bar{L}} \mathrm{Tr}(F_{\bar{A}}) \wedge \delta\varphi \end{aligned}$$

The first term vanishes since L is coassociative and the second term vanishes since in addition $\mathrm{Tr}(F_A)$ is self-dual. Now consider again the functional $\Phi_{4,\varphi}$, but consider also variations in φ. We have

$$d\Phi_{4,\varphi}|_{(\varphi,L,A)} (\delta\varphi, \delta L, \delta A) = \frac{i}{2\pi} \int_{\bar{L}} \mathrm{Tr}(F_{\bar{A}}) \wedge \delta\varphi$$

since the variations in L and A vanish because (L, A) is a coassociative cycle. Thus

$$\mu^*(\alpha) = -\frac{1}{2} d\Phi_{4,\varphi}$$

and hence as before $\mu^*(\omega) = 0$. ∎

As in the associative case, it would be interesting to be able to relate the non-degeneracy of the critical points of $\Phi_{4,\varphi}$ to the deformation theory. In constrast to the instanton case, in [31] McLean proved that the local deformations of coassociative submanifolds are always unobstructed. The moduli space of coassociative submanifolds is smooth and of dimension $b^2_-(L)$. On the other hand, a generic L does not support any self-dual connections of rank 1. They occur discretely on generic $b^2_-(L)$-dimensional families of metrics on L. Therefore, when such genericity occurs, then our above arguments really show that the image of Abel-Jacobi map is indeed a Lagrangian subspace in the universal intermediate Jacobian.

5.4 Universal moduli of Donaldson-Thomas connections

Finally, we consider the functional $\Phi_{k,\varphi}$ from (5.2) for $k = 7$. In this case $N = M$ necessarily, and thus we can only vary a connection A on a bundle E over M. We will denote the functional by $\Phi_{\mathcal{DT}}$ this time. Explictly,

$$\Phi_{\mathcal{DT}}(A) = \Phi_{7,\varphi}(M, A) = \int_{\bar{M}} \mathrm{Tr} \left(\frac{3}{6} \left(\frac{i}{2\pi} F_{\bar{A}} \right)^2 \wedge \pi^*(\psi) + \frac{1}{24} \left(\frac{i}{2\pi} F_{\bar{A}} \right)^4 \right)$$

where $\bar{M} = [0,1] \times M$ and $\bar{A} = \pi^*(A_t)$. We also have that

$$\Phi_{\mathcal{DT}}(A) = -\frac{1}{4\pi^2} \int_0^1 \int_M dt \wedge \operatorname{Tr}\left(A_t' \wedge \left(F_{A_t} \wedge \psi - \frac{1}{24\pi^2} F_{A_t}^3 \right) \right) \tag{5.18}$$

by using (5.3). The critical points of $\Phi_{\mathcal{DT}}$ are described by the following theorem.
Theorem 5.13. *The connection A_0 is a critical point of $\Phi_{\mathcal{DT}}$ if and only if A_0 satisfies the* deformed Donaldson-Thomas *equation:*

$$F_{A_0} \wedge \psi - \frac{1}{24\pi^2} F_{A_0}^3 = 0 \tag{5.19}$$

Proof. We will choose A_0 to be our base point. Consider a variation of the connection: $A_t = A_0 + tA$. Hence $F_{A_t} = F_{A_0} + t(\cdots) + t^2(\cdots)$. Using equation (5.19), (where $A_t' = A$) we have $\frac{\partial}{\partial t}\big|_{t=0} \Phi_{\mathcal{DT}}(A_t) =$

$$\begin{aligned} & \frac{\partial}{\partial t}\bigg|_{t=0} - \frac{1}{4\pi^2} \int_0^t \int_M ds \wedge \operatorname{Tr}\left(A \wedge \left(F_{A_0} \wedge \psi - \frac{1}{24\pi^2} F_{A_0}^3 + s(\cdots) \right) \right) \\ = \quad & -\frac{1}{4\pi^2} \int_M \operatorname{Tr}\left(A \wedge \left(F_{A_0} \wedge \psi - \frac{1}{24\pi^2} F_{A_0}^3 \right) \right) \end{aligned}$$

This must vanish for all possible A, and thus the theorem follows. □

Remark 5.14. In [9], Donaldson and Thomas introduced the equation

$$F_A \wedge \psi = 0$$

for a connection A on a G_2-manifold M, as a generalization of the *anti-self-dual* equations of Yang-Mills theory, to this setting. Such connections have curvature 2-form F_A contained in $\Omega^2_{14} \cong \mathfrak{g}_2$. The deformed Donaldson-Thomas equation was introduced in [29], where F_A is used instead of $\frac{i}{2\pi} F_A$. The cubic correction term allows the deformed DT equation to fit into the general setting described in Section 5.1. This 'deformed' equation has been studied by physicists in [6].

One can obtain the original Donaldson-Thomas connections from the same functional after truncating the Chern character terms. That is, replacing exp by $\exp_{\leq 2}(x) = 1 + x + \frac{1}{2}x^2$.
Definition 5.15. A connection A satisfying equation (5.19) will be called a *DDT connection.* Using the notation of (5.4) we see that $\mathcal{N}_{7,\varphi}$ is the space of DDT connections with respect to the G_2-structure φ.

Now consider the Abel-Jacobi map from Definition (5.3) with $k = 7$, and take $l = 4$. Let us denote ν_7^4 by χ. Explictly, we have

$$\chi(\varphi, A) = (\varphi, \bar{\chi}(A)) = (\varphi, C)$$

where $C = \bar{\chi}(A) \in H^3(M, S^1)$ is defined by

$$\int_M \bar{\chi}(A) \wedge D = \int_{M \times [0,1]} \left(r\, \pi^*(*\varphi) - \frac{1}{8\pi^2} \operatorname{Tr}(F_{A_t}^2) \right) \wedge \pi^*(D) \tag{5.20}$$

for any $D \in H^4(M, S^1)$, where r is the rank of E. Notice that the first term has no dt component, and thus must vanish on $\bar{M} = M \times [0,1]$. Then since $\frac{i}{2\pi} \operatorname{Tr}(F_{A_t})$ is an integral class, this map is well-defined.

Proposition 5.16. *The image of χ in $\mathcal{J}$ is isotropic with respect to the symplectic structure $\tilde{\omega}$ on $\mathcal{J} \cong \mathcal{M} \times H^3(M, S^1)$ described in* (4.13).

Proof. Proceeding as in the proof of Proposition 5.7, it is easy to check that

$$\chi^*(\tilde{\alpha}) = d\Phi_{7,\varphi}$$

where $\tilde{\omega} = d\tilde{\alpha}$ is the symplectic form. One needs to use the fact that $(X \lrcorner \psi) \wedge \psi$ vanishes for any vector field X, and that an 8-form on $\bar{M} = M \times [0,1]$ vanishes unless it has a dt factor. The details are left to the reader. □

As before, the non-degeneracy of the critical points of $\Phi_{7,\varphi}$ may be related to the deformation theory of deformed Donaldson-Thomas connections. As far as we know this has not been studied.

5.5 Universal Moduli as Minimizers

Now we consider another functional Ψ which looks very similar to Φ, but is slightly different. It is defined as follows.

$$\begin{aligned} \Psi_{k,\varphi} &: \widetilde{\mathcal{C}_k} \to \mathbb{R} \\ \Psi_{k,\varphi}(N, A) &= \int_N \mathrm{Tr}\left[\exp\left(\frac{i}{2\pi} F_A + \varphi + *\varphi\right)\right] \end{aligned}$$

We also write $\Psi_k(\varphi, N, A) = \Psi_{k,\varphi}(N, A)$. Note that since each N in our configuration space is in the same homology class, $\Psi_{k,\varphi}$ is a *topological number*. Let $\mathcal{YM}(A)$ denote the Yang-Mills functional of the connection A. The *size* of a pair (N, A) in $\widetilde{\mathcal{C}_k}$ is defined to be $\mathsf{vol}(N) + \mathcal{YM}(A)$.

Suppose that $k = 3$. Then we have

$$\Psi_{3,\varphi}(N, A) = \int_N \varphi \le \mathsf{vol}(N) + \mathcal{YM}(A)$$

and the minimum is attained exactly along the critical points of $\Phi_{3,\varphi}$, namely associative submanifolds coupled with flat connections.

When $k = 4$, we need to assume that $\mathrm{c}_2(E) < 0$, so that the absolute minima of $\mathcal{YM}(A)$ are the self-dual connections. Then

$$\Psi_{4,\varphi}(L, A) = \int_L \left(*\varphi - \frac{1}{8\pi^2}\mathrm{Tr}\left(F_A^2\right)\right) \le \mathsf{vol}(L) + \mathcal{YM}(A)$$

and the minimum is attained along the coassociative submanifolds coupled with self-dual connections over them.

Finally when $k = 7$, that is when $N = M$, we have

$$\Psi_{7,\varphi}(M, A) = \int_M \left(\varphi \wedge *\varphi - \frac{1}{8\pi^2}\mathrm{Tr}\left(F_A^2\right) \wedge \varphi\right) \le \mathsf{vol}(M) + \mathcal{YM}(A)$$

and here the minimum is attained for Donaldson-Thomas connections over M, as explained in the section on Yang-Mills calibrations in [27].

Therefore we see that this topological number $\Psi_{k,\varphi}$ in each case serves as an absolute lower bound for the size of (N, A), and this lower bound is attained exactly at the critical points of the functional $\Phi_{k,\varphi}$. (This statement requires the rank $r = 1$ for the second case, and we are using ordinary Donaldson-Thomas connections here instead of DDT connections for the third case.)

In this last case, if we also allow the 3-form φ to vary in a fixed cohomology class of M, then the first term is the Hitchin functional defined in [15], and the critical points, which are also minimum points, are precisely given by torsion free G_2-structures on M. Quantizing the Hitchin functional has been discussed in many physics papers, including [2] and [7]. Therefore, it is reasonable to expect that one could obtain interesting physics by quantizing our functional as well.

6 Conclusion

We close by considering generalizations of our constructions as well as questions for future studies.

6.1 Generalizations

In this paper, we have shown that the universal intermediate Jacobian $\mathcal{J}$ of a G_2-manifold has a natural pseudo-Kähler structure and it admits many Lagrangian subspaces given by the universal moduli of associative cycles, coassociative cycles, and deformed Donaldson-Thomas connections. All these are critical points for our functional Φ.

For any given torsion-free G_2-structure φ, its intermediate Jacobian $\mathcal{J}_\varphi$ is also a Lagrangian submanifold of $\mathcal{J}$ and together they form a Lagrangian fibration over $\mathcal{M}$. We can also replace $\mathcal{J}_\varphi$ by the moduli space of flat 2-gerbes on M and obtain similar pictures. A good reference for gerbes, from a differential-geometric point of view, is Hitchin [14].

In fact all of these Lagrangian submanifolds $\mathcal{J}_\varphi \subset \mathcal{J}$ can also be described as critical points of our functional Φ, if we replace $F_{\bar{A}}$ by the curvature of a gerbe as follows. Let E be a $U(1)$ 2-gerbe on a manifold $\bar{N}$. A 2-connection on E is locally given by a 3-form C on M and its 2-curvature $F_C = dC$ is a well-defined closed 4-form on $\bar{N}$. The cohomology class $c(E) = \left[\frac{i}{2\pi} F_C\right] \in H^4(\bar{N}, \mathbb{R})$ is independent of the choice of C, which is analogous to the first Chern class of a complex line bundle.

Consider $k = 7$, and let us fix a $U(1)$ 2-gerbe E on M together with a fixed 2-gerbe connection C_0 on it. We define a functional on the space of 2-gerbe connections on E, and we denote this configuration space by $\mathcal{C}$ again.

$$\begin{aligned}
\Phi_{7,\varphi} &: \widetilde{\mathcal{C}} \to \mathbb{R} \\
\Phi_{7,\varphi}(A) &= \int_{\bar{N}} \exp\left(\frac{i}{2\pi} F_{\bar{C}} + \varphi + *\varphi\right) \\
&= \frac{1}{2}\int_{\bar{N}} \left(\frac{i}{2\pi} F_{\bar{C}} + *\varphi\right)^2
\end{aligned}$$

where $\bar{N} = M \times [0,1]$ and the 2-connection $\bar{C}$ over it is determined by a path of 2-connections C_t over M joining C_0 and C.

As before, the Euler-Lagrangian equation for $\Phi_{7,\varphi}$ is given by

$$\frac{i}{2\pi} F_{\bar{C}} + *\varphi = 0$$

Hence the moduli space of solutions $\mathcal{N}_{7,\varphi}$ is non-empty if and only if $c(E) = -[*\varphi] \in H^4(M, \mathbb{R})$, in which case $\mathcal{N}_{7,\varphi}$ is isomorphic to the space of flat 2-gerbes over M, which is $H^3(M, S^1)$. Furthermore this equals the union of moduli $\mathcal{N}_7$. Thus this space $\mathcal{N}_7$ coincides with the Lagrangian fibers of the universal intermediate Jacobian $\mathcal{J} \to \mathcal{M}$.

6.2 Questions for future study

It would be interesting to see explicitly how these constructions relate to the analgous ones in the case of a Calabi-Yau 3-fold. Recall that the universal intermediate Jacobian $\mathcal{J}$ in the Calabi-Yau 3-fold case is *hyperKähler*, thus it admits a whole S^2 family of complex structures. In the G_2 case these reduce to just one complex structure. While it is true that $M^7 = X^6 \times S^1$ admits a torsion-free G_2-structure when X^6 is a Calabi-Yau 3-fold, the precise relationship between the moduli spaces of these structures is more complicated. This is because the complex and Kähler moduli of X^6 mix together in a non-trivial way to yield the G_2-moduli of M^7. This is reflected in the fact that in this case, $b_1(M) = 1$, and most of the results obtained in this paper were specific to the case $b_1(M) = 0$, which corresponds to full holonomy G_2.

Two more difficult questions which need to be pursued are the following:

- What is the intersection theory of all these Lagrangian subspaces in the universal intermediate G_2-Jacobian $\mathcal{J}$?
- What is the relation of these constructions with the *topological quantum field theory* (TQFT) of Calabi-Yau 3-folds and G_2-manifolds, as described, for example, in [25]?

We hope to address these questions in future work.

Another situation that should be studied is when the G_2-manifold is non-compact, with an asymptotically cylindrical end, asymptotic to a cylinder on a Calabi-Yau 3-fold. Then the corresponding constructions on the CY 3-fold give the hyperKähler universal intermediate Jacobian, and various complex Lagrangian submanifolds inside of it. (See [23, 24] for more about asymptotically cylindrical Calabi-Yau 3-folds and G_2 manifolds.) The relation between the two constructions in this case needs to be understood. In addition the Topological Field Theory picture of such a G_2/CY3 situation should be part of Topological M-Theory. Similarly one could consider a G_2-manifold which is the end of a Spin(7)-manifold with an asymptotically cylindrical end. Then the G_2/Spin(7) story should be an analogue/generalization of the Chern-Simons/Donaldson Topological Field Theory story.

There also remains much to understand about the differential geometry of the moduli space $\mathcal{M}$ of torsion-free G_2-structures. In particular, the Hitchin metric $\mathcal{G}$ deserves more attention. Theorem 3.10, our main result relating the Yukawa coupling to the superpotential function f, is used in [21] to study the *curvature* of this metric.

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